Twin Primes Theorem

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Abstract

We define a twin prime pair (p,q) of primes p,q if $q-p=2$. We call q an upper twin prime. Then we prove,

$$\sum^{*}\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$$

is the exact count of the number of upper twin primes in the interval (p_θ, N) with N in the interval $p_\theta < N \le p_{\theta+1}^2$. The variables and the sum symbol $\sum^{*}$ mean,

1. N is an even natural number greater than 3^2.
2. p_θ is the largest prime less than $\sqrt{N}$,
3. $p_{\theta+1}$ is the next largest prime,
4. $[x]$ is the greatest integer function,
5. d_1 is the product of one or more elements of the set $\{2,3,\ldots,p_\theta\}\cup\{1\}$, where $2,3,5,\ldots,p_\theta$ are consecutive primes.
6. d_2 is the product of one or more elements of the set $\{3,5\ldots,p_\theta\}\cup\{1\}$ where $3,5,\ldots,p_\theta$ are consecutive primes That is $gcd(d_2,2)=1$ or $2\nmid d_2$.
7. $gcd(d_1,d_2)=1$.
8. The sum is over all possible values of d_1 and d_2.
9. $\mu(d)$ is the Möbius function.
10. x_1 is the least non-negative solution of the system of simultaneous linear congruences,
$$x\equiv 0(\text{mod } d_1), x\equiv 2(\text{mod } d_2)$$

This formula also counts the number 1 so we need to make that correction.

Using the formula we then prove there are an infinite number of twin prime pairs. We do this by assuming there is a greatest twin prime pair and then obtaining a contradiction.

Contents

Chapter 1

Setting up and Heuristic Evidence

1.1 The Primod-p_θ Number System

Notation 1. *The symbol p_θ means the prime number θ in order of magnitude. For example, p_5 is the 5th prime number, namely 11.*

The Primod–p_θ number system, with p_θ any prime number, is based on a different concept to decimal and binary number systems. The primod number, or simply primod, of a positive integer n is written as $a\ b\ c\ d\ \ldots$ where it is assumed the consecutive prime numbers $2, 3, 5, 7, \ldots, p_\theta$ are written from left to right across the top of each digit which is calculated as the least non-negative value of $x \equiv n(\mod\ p)$ where $p \leq p_\theta$ is the relevant prime. The term primod has been chosen as the abbreviated combination of prime and modulus since each digit is calculated from the least non-negative modulus of the integer to the respective prime.

Example 1. *In the Primod-11 number system, the primod* 1 2 4 6 9 *is interpreted as:*

$$\begin{array}{ccccc} 2 & 3 & 5 & 7 & 11 \\ 1 & 2 & 4 & 6 & 9 \end{array}$$

or the least non-negative (lnn) integer x that satisfies the system of simultaneous congruences :

$$x \equiv 1(\text{mod } 2),\ x \equiv 2(\text{mod } 3),\ x \equiv 4(\text{mod } 5),\ x \equiv 6(\text{mod } 7),\ x \equiv 9(\text{mod } 11)$$

or 2099 in decimal notation.

We shall refer to the digits of a primod as p-digits, e.g., the 2-digit of the primod of the integer 5 is 1 and its 3-digit is 2. We note that the p-digit of any integer is one of $\{0, 1, \ldots, p-1\}$. Thus the p–digits of 5 can only be one of $\{0, 1, 2, 3, 4\}$.

To get a better sense of the primod number system, let's consider the Primod-11 number system, the set of primods of the integers n such that $1 \leq n \leq 2 \times 3 \times 5 \times 7 \times 11$.

There are 2310 primods in this set. The primods of integers 1-171 are given in Table 1 below. The simple test for a number to be a prime is to show it is not divisible by any prime less than its square root. Hence the shaded integers in Table 1 with no 0-digits that are less than $169 = 13^2$, and not just $121 = 11^2$, are all primes.
In general in the Primod-p_θ number system all integers less than $p_{\theta+1}^2$ with no zero $p-$digits are all primes, $p_{\theta+1}$ being the next largest prime to p_θ.
The numbers 2,3,5,7,11 that "define" the Primod-11 number system are excluded since they all have a zero digit occasioned by $p \equiv 0(\text{mod } p)$. They are, however, primes. We call them the sieving primes.

1.2 Notation and Definitions

Notation 2. $T(p_\theta, N, M)$
With reference to the Primod-p_θ number system, if N is any positive integer and M any non-negative even integer, $T(p_\theta, N, M)$ will denote the number of non-negative integers less than N with primods satisfying the condition that none of their p-digits is either 0 or the same as the corresponding p-digit of the primod of the integer M.

Definition 1. *Twin primes*
If q_1, q_2 are primes such that $q_2 - q_1 = 2$ then we say (q_2, q_1) is a twin prime pair. We say such a q_2 is an upper twin prime.

Our first goal is to develop a formula for counting the number of upper twin primes in a given interval.

1.3 Primods with no p-digit equal to 0 or $2(\text{ mod } p)$

Let us now consider, still for the primod-11 number system and referencing Table 1, primods with no p-digit equal to either 0 or 2, with $p \in \{2, 3, 5, 7, 11\}$. We shall denote these as the allowable primods.
The respective integers are among those highlighted in Table 1 and are the sets,

$$\{1\} \cup \{19, 31, 43, 61, 73, 103, 109, 139, 151\}$$

The single element in the first set is the number 1 which is not a prime. The second set consists of the primods of integers greater than 2. The elements are the primods of primes if the related integer is less than 169 and may or may not be primes otherwise. Now the elements of the second set are the integers, q_2, with allowable primods having no $p-$digit equal to 0 or 2. Therefore the primod of the integer q_1 such that $q_1 = q_2 - 2$ cannot contain any $p-$digits equal to 0 so q_1 must be a prime and (q_1, q_2) is a twin prime pair. For example $19 = 11458$ and $19 - 2 = 12236$ which is the primod of 17 which must be a prime and $(17, 19)$ a twin prime pair.

	2	3	5	7	11		2	3	5	7	11		2	3	5	7	11		2	3	5	7	11
						43	1	1	3	1	10	86	0	2	1	2	9	129	1	0	4	3	8
1	1	1	1	1	1	44	0	2	4	2	0	87	1	0	2	3	10	130	0	1	0	4	9
2	0	2	2	2	2	45	1	0	0	3	1	88	0	1	3	4	0	**131**	1	2	1	5	10
3	1	0	3	3	3	46	0	1	1	4	2	**89**	1	2	4	5	1	132	0	0	2	6	0
4	0	1	4	4	4	**47**	1	2	2	5	3	90	0	0	0	6	2	133	1	1	3	0	1
5	1	2	0	5	5	48	0	0	3	6	4	91	1	1	1	0	3	134	0	2	4	1	2
6	0	0	1	6	6	49	1	1	4	0	5	92	0	2	2	1	4	135	1	0	0	2	3
7	1	1	2	0	7	50	0	2	0	1	6	93	1	0	3	2	5	136	0	1	1	3	4
8	0	2	3	1	8	51	1	0	1	2	7	94	0	1	4	3	6	**137**	1	2	2	4	5
9	1	0	4	2	9	52	0	1	2	3	8	95	1	2	0	4	7	138	0	0	3	5	6
10	0	1	0	3	10	**53**	1	2	3	4	9	96	0	0	1	5	8	**139**	1	1	4	6	7
11	1	2	1	4	0	54	0	0	4	5	10	**97**	1	1	2	6	9	140	0	2	0	0	8
12	0	0	2	5	1	55	1	1	0	6	0	98	0	2	3	0	10	141	1	0	1	1	9
13	1	1	3	6	2	56	0	2	1	0	1	99	1	0	4	1	0	142	0	1	2	2	10
14	0	2	4	0	3	57	1	0	2	1	2	100	0	1	0	2	1	143	1	2	3	3	0
15	1	0	0	1	4	58	0	1	3	2	3	**101**	1	2	1	3	2	144	0	0	4	4	1
16	0	1	1	2	5	**59**	1	2	4	3	4	102	0	0	2	4	3	145	1	1	0	5	2
17	1	2	2	3	6	60	0	0	0	4	5	**103**	1	1	3	5	4	146	0	2	1	6	3
18	0	0	3	4	7	**61**	1	1	1	5	6	104	0	2	4	6	5	147	1	0	2	0	4
19	1	1	4	5	8	62	0	2	2	6	7	105	1	0	0	0	6	148	0	1	3	1	5
20	0	2	0	6	9	63	1	0	3	0	8	106	0	1	1	1	7	**149**	1	2	4	2	6
21	1	0	1	0	10	64	0	1	4	1	9	**107**	1	2	2	2	8	150	0	0	0	3	7
22	0	1	2	1	0	65	1	2	0	2	10	108	0	0	3	3	9	**151**	1	1	1	4	8
23	1	2	3	2	1	66	0	0	1	3	0	**109**	1	1	4	4	10	152	0	2	2	5	9
24	0	0	4	3	2	**67**	1	1	2	4	1	110	0	2	0	5	0	153	1	0	3	6	10
25	1	1	0	4	3	68	0	2	3	5	2	111	1	0	1	6	1	154	0	1	4	0	0
26	0	2	1	5	4	69	1	0	4	6	3	112	0	1	2	0	2	155	1	2	0	1	1
27	1	0	2	6	5	70	0	1	0	0	4	**113**	1	2	3	1	3	156	0	0	1	2	2
28	0	1	3	0	6	**71**	1	2	1	1	5	114	0	0	4	2	4	**157**	1	1	2	3	3
29	1	2	4	1	7	72	0	0	2	2	6	115	1	1	0	3	5	158	0	2	3	4	4
30	0	0	0	2	8	**73**	1	1	3	3	7	116	0	2	1	4	6	159	1	0	4	5	5
31	1	1	1	3	9	74	0	2	4	4	8	117	1	0	2	5	7	160	0	1	0	6	6
32	0	2	2	4	10	75	1	0	0	5	9	118	0	1	3	6	8	161	1	2	1	0	7
33	1	0	3	5	0	76	0	1	1	6	10	119	1	2	4	0	9	162	0	0	2	1	8
34	0	1	4	6	1	77	1	2	2	0	0	120	0	0	0	1	10	**163**	1	1	3	2	9
35	1	2	0	0	2	78	0	0	3	1	1	121	1	1	1	2	0	164	0	2	4	3	10
36	0	0	1	1	3	**79**	1	1	4	2	2	122	0	2	2	3	1	165	1	0	0	4	0
37	1	1	2	2	4	80	0	2	0	3	3	123	1	0	3	4	2	166	0	1	1	5	1
38	0	2	3	3	5	81	1	0	1	4	4	124	0	1	4	5	3	**167**	1	2	2	6	2
39	1	0	4	4	6	82	0	1	2	5	5	125	1	2	0	6	4	168	0	0	3	0	3
40	0	1	0	5	7	**83**	1	2	3	6	6	126	0	0	1	0	5	**169**	1	1	4	1	4
41	1	2	1	6	8	84	0	0	4	0	7	**127**	1	1	2	1	6	170	0	2	0	2	5
42	0	0	2	0	9	85	1	1	0	1	8	128	0	2	3	2	7	171	1	0	1	3	6

Table 1

Now if no $p-$digit can equal 0 or 2 then the $2-$digit of any integer can only be $2-1=1$ and each other $p-$digit can only have $p-2$ values. There are therefore,

$$(2-1)\times(3-2)\times(5-2)\times(7-2)\times(11-2)=135$$

allowable primods in the Primod-11 number system that satisfy the specified conditions.

Then, we predict there are,

$$T(11,169,2)\approx\frac{2-1}{2}\times\frac{3-2}{3}\times\frac{5-2}{5}\times\frac{7-2}{7}\times\frac{11-2}{11}\times 169\approx 10,$$

primods of integers less than 169 satisfying the given conditions.

This includes the primod of the number 1, assigned to the first set above. In the second set, the primods of the integers greater than 2, we would therefore expect about 9 primes between 11 and 168 that are the larger members of a twin prime pair. There are exactly 9, namely,

$$19,31,43,61,73,103,109,139,151$$

In other words there are 9 twin prime pairs between 11 and 169 inclusive, namely,

$$(17,19),(29,31),(41,43),(59,61),(71,73),(101,103),(107,109),(137,139),(149,151)$$

The easy extension of the above to the general case is given by,

$$\begin{aligned}T(p_\theta,N,2)&\approx\frac{2-1}{2}\times\frac{3-2}{3}\times\frac{5-2}{5}\times\frac{7-2}{7}\times\frac{11-2}{11}\times\cdots\times\frac{p_\theta-2}{p_\theta}\times N\\&=\left(1-\frac{1}{2}\right)\prod_{p=3}^{p_\theta}\left(1-\frac{2}{p}\right)\times N\end{aligned}$$

If we choose $N : p_\theta < N < p_{\theta+1}^2$ where p_θ and $p_{\theta+1}$ are successive primes, then $T(p_\theta,N,2)-1$ is the predicted number of twin prime pairs with the upper prime greater than p_θ and less than N. Since $T(p_\theta,N,2)$ also counts the number 1, our goal is to prove $T(p_\theta,N,2)$ is always greater than 1.

Now, choosing $N=p_\theta^2$,

$$\begin{aligned}T(p_\theta,N,2)&\approx\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\frac{5}{7}\times\frac{9}{11}\times\frac{15}{17}\cdots\times\frac{p_\theta-2}{p_\theta}\times N\\&>\frac{1}{2}\times\frac{1}{3}\times\frac{3}{5}\times\frac{5}{7}\times\frac{7}{9}\times\frac{9}{11}\times\frac{11}{13}\times\cdots\times\frac{p_\theta-4}{p_\theta-2}\times\frac{p_\theta-2}{p_\theta}\times p_\theta^2\\&>\frac{p_\theta}{2}\end{aligned}$$

This predicts there are always many more than $\frac{p_\theta}{2}$ twin prime pairs between p_θ and $N=p_\theta^2$ where p_θ is the largest prime less than $\sqrt{N}$.

Chapter 2

Counting Allowable Primods

2.1 Finding the Upper Twin Primes

Twin primes are pairs of primes q_1, q_2 satisfying $q_2 - q_1 = 2$. In the primod$-p_\theta$ number system, any prime q_2 such that $p_\theta < q_2 < p_{\theta+1}^2$ is the upper twin of a twin prime pair provided the primod of q_2 has no $p-$digit equal to 2 for $p \in \{2, 3, \ldots, p_\theta\}$, since if any $p-$digit of q_2 equals 2, then the corresponding p-digit of q_1 is 0, meaning q_1 is not a prime.

For example, the primod 1 1 2 5 8 may be the primod of a prime, having no zero digits, but it cannot be the upper prime of a twin prime pair, since the lower member, with primod 1 2 0 3 6 (after subtracting 2 from each $p-$digit,) has a 5$-$digit of 0 so is divisible by 5.

Since $T(p_\theta, N, 2)$ counts only primods with no $p-$digit equal to 0 or 2 then, for $p_\theta \leq N < p_{\theta+1}^2$, we have that $T(p_\theta, N, 2)$ is the count of the upper primes of all twin prime pairs with upper primes less than N and greater than p_θ.

We begin with finding the upper twin primes between 5 and 32.

Consider $T(5, 32, 2)$ in the Primod$-$5 number system, which is the count of the primods of upper members of twin prime pairs between 5 and 32. To identify the integers with allowable primods, we need to sieve out the integers with primods having any digits equal to either 0 or 2. That means we need to sieve out the integers x of any of the forms,

$$x \equiv 0 (\text{mod } 2), x \equiv 0 (\text{mod } 3), x \equiv 0 (\text{mod } 5)$$

x/p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2		2
3			3			3			3			3			3			3			3			3			3			3	3	
5					5					5					5					5					5					5		
A	1						7				11		13				17		19				23						29		31	
3’		3’			3’			3’			3’			3’			3’			3’			3’			3’			3’			3’
5’		5’					5’					5’					5’					5’					5’					5’
B	1												13						19												31	

Table 2

as well as the numbers satisfying,

$$x \equiv 2(\text{mod } 2), x \equiv 2(\text{mod } 3), x \equiv 2(\text{mod } 5)$$

We first note that the integers satisfying $x \equiv 0(\text{mod } 2)$ are the same as those satisfying $x \equiv 2(\text{mod } 2)$ so we need to remove one of these conditions to avoid double counting or deletions. We omit $x \equiv 2 (\bmod\ 2)$. Then the numbers we are deleting via the sieve are:

$$\begin{aligned}
x &\equiv 0(\bmod\ 2): \ 2, 2+2, 2+2+2, \ldots \\
x &\equiv 0(\bmod\ 3): \ 3, 3+3, 3+3+3, \ldots \\
x &\equiv 0(\bmod\ 5): \ 5, 5+5, 5+5+5 \ldots \\
x &\equiv 2(\bmod\ 3): \ 2, 2+3, 2+3+3, \ldots \\
x &\equiv 2(\bmod\ 5): \ 2, 2+5, 2+5+5 \ldots
\end{aligned}$$

A double Erasthosthenes type sieve can be set up to generate the allowable primods with no p–digits equal to 0 or 2. In Table 2 above, the numbers 1 to 32 are placed on the top row.
The first sieve, indicated by the numbers 2,3,5 in the first column removes all numbers x such that,

$$x \equiv 0(\bmod\ 2),\ x \equiv 0(\bmod\ 3),\ x \equiv 0(\bmod 5),$$

and we fill in each row with a “2” at 2,2+2,2+2+2,...Similarly for rows with 3 and 5. This sieves out the number 1 as well as the integers between 1 and 32 with non-allowable primods as indicated by the shaded squares on the A-row. We are left with the set of primes between 5 and 32 as well as the number 1, namely

$$1, 7, 11, 13, 17, 19, 23, 29, 31$$

The second sieve, indicated by the numbers 3’ and 5’ in the first column, sieves out all numbers x such that,

$$\begin{aligned}
x &\equiv 2(\bmod\ 3),\ \text{namely } 2, 5, 8, \cdots \\
x &\equiv 2(\bmod\ 5),\ \text{namely } 2, 7, 12, \cdots
\end{aligned}$$

The numbers (which must also be primes) sieving through onto the bottom B-row have primods with no p–digits equal to 0 or 2. With the exception of the number 1, they are the upper primes of the twin prime pairs between 5 and 32, namely $\{13, 19, 31\}$.

2.2 Counting the Upper Twin Primes

Let us now count $T(5, 32, 2)$ or the number of integers between 1 and 32 inclusive with primods having no p–digit equal to 0 or 2, that is the number of upper twin primes of a twin prime pair. We need Tables 3 and 4 below. Table 3 shows how we set up the count.
We begin by taking a set of integers 1 to N such as 1 to 32 and noting 5 is the greatest prime less than $\sqrt{32}$, so we have the sieving primes 2,3,5. For the top row of numbers

in Table 3 we note $x = 0$ is the least non-negative solution of the pair of simultanious congruences,

$$x \equiv 0(\bmod 1),\ x \equiv 2(\bmod 1)$$

and each other solution is obtained by simply repeatedy adding 1.
The counting argument is we delete according to the products (including the number 1) of an odd number of the sieving primes and we replace according to an even number. Hence we delete all $x \in \{1, 2, 3 \dots 32\}$ where $x \equiv 0(\bmod 2)$ but we express the first deletion as the least non-negative solution to a pair of simultaneous congruences, namely,

$$x \equiv 0(\bmod 2),\ x \equiv 2(\bmod 1)$$

Similarly we solve the systems,

$$x \equiv 0(\bmod 3),\ x \equiv 2(\bmod 1)$$
$$x \equiv 0(\bmod 5),\ x \equiv 2(\bmod 1)$$
$$x \equiv 2(\bmod 3),\ x \equiv 2(\bmod 1)$$
$$x \equiv 2(\bmod 5),\ x \equiv 2(\bmod 1)$$

The least non-negative solutions to these pairs of simultaneous congruences are given in the x_1 column of Table 3.
We are solving systems of the form $x \equiv 0(\bmod d_1),\ x \equiv 2(\bmod d_2)$ with the values of d_1 and d_2 given in the first two columns of Table 3.

We identify the locations of double deletions by solving,

$$x \equiv 0(\bmod 2),\ x \equiv 2(\bmod 3)$$
$$x \equiv 0(\bmod 2),\ x \equiv 2(\bmod 5)$$
$$x \equiv 0(\bmod 3),\ x \equiv 2(\bmod 5)$$
$$x \equiv 0(\bmod 5),\ x \equiv 2(\bmod 3)$$

But we also have double deletions with respect to,

1. $x \equiv 0(\bmod 2),\ x \equiv 0(\bmod 3) \Leftrightarrow x \equiv 0(\bmod 2 \times 3),\ x \equiv 2(\bmod 1)$

2. $x \equiv 0(\bmod 2),\ x \equiv 0(\bmod 5) \Leftrightarrow x \equiv 0(\bmod 2 \times 5),\ x \equiv 2(\bmod 1)$

3. $x \equiv 0(\bmod 3),\ x \equiv 0(\bmod 5) \Leftrightarrow x \equiv 0(\bmod 3 \times 5),\ x \equiv 2(\bmod 1)$

4. $x \equiv 2(\bmod 3),\ x \equiv 2(\bmod 5) \Leftrightarrow x \equiv 0(\bmod 3 \times 5),\ x \equiv 2(\bmod 1)$

In each of the above we replace a pair of simultaneous congruences with an equivalent pair, so we are always solving systems of the form $x \equiv 0(\bmod d_1)$ and $x \equiv 2(\bmod d_2)$

d_1	d_2	*System* $x \equiv 0(\text{mod } d_1)$ $x \equiv 2(\text{mod } d_2)$	x_1	$d_1 d_2$
1	1	$x \equiv 0(\text{mod } 1)$ $x \equiv 2(\text{mod } 1)$	0	1
2	1	$x \equiv 0(\text{mod } 2)$ $x \equiv 2(\text{mod } 1)$	0	2
3	1	$x \equiv 0(\text{mod } 3)$ $x \equiv 2(\text{mod } 1)$	0	3
1	3	$x \equiv 0(\text{mod } 1)$ $x \equiv 2(\text{mod } 3)$	2	3
2×3	1	$x \equiv 0(\text{mod } 6)$ $x \equiv 2(\text{mod } 1)$	0	6
2	3	$x \equiv 0(\text{mod } 2)$ $x \equiv 2(\text{mod } 3)$	2	6
5	1	$x \equiv 0(\text{mod } 5)$ $x \equiv 2(\text{mod } 1)$	0	5
1	5	$x \equiv 0(\text{mod } 1)$ $x \equiv 2(\text{mod } 5)$	2	5
2×5	1	$x \equiv 0(\text{mod } 10)$ $x \equiv 2(\text{mod } 1)$	0	10
2	5	$x \equiv 0(\text{mod } 2)$ $x \equiv 2(\text{mod } 5)$	2	10
3	5	$x \equiv 0(\text{mod } 3)$ $x \equiv 2(\text{mod } 5)$	12	15
5	3	$x \equiv 0(\text{mod } 5)$ $x \equiv 2(\text{mod } 3)$	5	15
2×3	5	$x \equiv 0(\text{mod } 6)$ $x \equiv 2(\text{mod } 5)$	12	30
2×5	3	$x \equiv 0(\text{mod } 10)$ $x \equiv 2(\text{mod } 3)$	20	30
3×5	1	$x \equiv 0(\text{mod } 15)$ $x \equiv 2(\text{mod } 1)$	0	15
1	3×5	$x \equiv 0(\text{mod } 1)$ $x \equiv 2(\text{mod } 15)$	2	15
$2 \times 3 \times 5$	1	$x \equiv 0(\text{mod } 30)$ $x \equiv 2(\text{mod } 1)$	0	30
2	3×5	$x \equiv 0(\text{mod } 2)$ $x \equiv 2(\text{mod } 15)$	2	30

Table 3

with the complete set of (d_1, d_2) values given in the first two columns of Table 3. Finally we have deletions due to,

$$x \equiv 0(\text{mod } 2),\ x \equiv 0(\text{mod } 3),\ x \equiv 0(\text{mod } 5) \Leftrightarrow x \equiv 0(\text{mod } 2 \times 3 \times 5),\ x \equiv 2(\text{mod } 1)$$

The least non-negative solution of these 18 pairs of simultaneous congruences are given in the x_1 column of Table 3.
We can summarize the above by saying we choose the product d_1 of one or more elements of the set $\{2, 3, 5\} \cup \{1\}$ and another product d_2 of one or more elements of the set $\{3, 5\} \cup \{1\}$ with $gcd(d_1, d_2) = 1$. We then solve all possible pairs of simultaneous congruences of the form,

$$x \equiv 0(\text{mod } d_1),\ x \equiv 2(\text{mod } d_2)$$

to find the values of x_1.
We now consider Table 4 where the x_1 and d_1, d_2 values have been transferred from Table 3. We are dealing with the same value, $N = 32$ and the same sieving primes $\{2, 3, 5\}$. Table 4 has been set up as two top lines and then four "quartets" each formed from a unique choice of d_1 and d_2 from products of one of more numbers chosen from $\{1, 3, 5\}$ with $gcd(d_1, d_2) = 1$.

d_1	d_2	x_1	d_1d_2	$\mu(d_1d_2) \times \left[\frac{32-x_1}{d_1d_2}\right]$	SR	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	1	0	1	32	32	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1
2	1	0	2	-16	-16		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1		−1
3	1	0	3	-10	-10			−1			−1			−1			−1			−1			−1			−1			−1			−1			−1		
1	3	2	3	-10	-11		−1			−1			−1			−1			−1			−1			−1			−1			−1			−1			−1
2 × 3	1	0	6	+5	5						+1						+1						+1						+1						+1		
2	3	2	6	+5	6		+1						+1						+1						+1						+1						+1
5	1	0	5	-6	-6					−1					−1					−1					−1					−1					−1		
1	5	2	5	-6	-7		−1					−1					−1					−1					−1					−1					−1
2 × 5	1	0	10	+3	3										+1										+1										+1		
2	5	2	10	+3	4		+1										+1										+1										+1
5	3	5	15	+1	2					+1															+1												
3	5	12	15	+1	2												+1															+1					
2×5	3	20	30	0	-1																				−1												
2 × 3	5	12	30	0	-1												−1																				
5 × 3	1	0	15	+2	2															+1															+1		
1	5 × 3	2	15	+2	3		+1															+1															+1
2 × 5 × 3	1	0	30	-1	-1																														−1		
2	5 × 3	2	30	-1	-2		−1																														−1
				4	4	1												1						1												1	

Table 4

In each quartet we observe the rule that the product of an odd number of primes is used for additions of $+1's$ and, in the same quartet the product of an even number of primes is used for deletions via $-1's$.
The method of inserting $+1s$ and $-1's$ is to calculate the four values of x_1 given by solving $x \equiv 0(\text{mod } d_1)$ and $x \equiv 2(\text{mod } d_2)$ for each choice of d_1 and d_2 and to insert a +1 at the positions $x_1,\ x_1+d_1d_2,\ x_1+2d_1d_2,\ \ldots$ where d_1d_2 is the product of and odd number of primes for which we have additions and a -1 at the positions $x_1,\ x_1+2d_1d_2,\ x_1+4d_1d_2,\ldots$ where d_1d_2 is the product of an even number of primes for which we have deletions.
The SR column gives the Sum of the $\pm 1's$ in each Row and its overall total in this case is 4 which is the correct count of the number of upper twin primes between 5 and 32 as well as the number 1.
Utilizing the Möbius function we can conjecture that the number of +1's or -1's in each row is given by $\mu(d_1d_2)\left[\frac{32-x_1}{d_1d_2}\right]$ and these results are shown in the column headed $\mu(d_1d_2)\left[\frac{32-x_1}{d_1d_2}\right]$.
We note that the bottom line sums of the $\mu(d_1d_2)\left[\frac{32-x_1}{d_1d_2}\right]$ and SR columns are the same in this case, namely 4, that being the number of upper twin primes between 5 and 32 as well as the number 1.
While the overall sums of the $\mu(d_1,d_2)\left[\frac{32-x_1}{d_1d_2}\right]$ and SR columns are both 4, we note the sum SR of each individual row does not always equal $\mu(d_1,d_2)\left[\frac{32-x_1}{d_1d_2}\right]$ but is sometimes $\mu(d_1,d_2)\left[\frac{32-x_1}{d_1d_2}\right] \pm 1$.Thus the (d_1,d_2) rows of $(3,1)$ and $(1,3)$ are,

d_1	d_2	x_1	d_1d_2	$\mu(d_1d_2)\left[\frac{32-x_1}{d_1d_2}\right]$	SR
3	1	0	3	-10	-10
1	3	2	3	-10	-11

But we also have the (d_1,d_2) rows of $(2\times 3,1)$ and $(2,3)$ thus,

d_1	d_2	x_1	d_1d_2	$\mu(d_1d_2)\left[\frac{32-x_1}{d_1d_2}\right]$	SR
2×3	1	0	6	5	5
2	3	2	6	5	6

Therefore the sum of this "quartet" of terms in $\sum \mu(d_1d_2)\left[\frac{32-x_1}{d_1d_2}\right]$ is the same as the corresponding four SR sums, that is, $-10-10+5+5=-10-11+5+6=-10$ and

hence each SR sum may be replaced by the corresponding $\mu(d_1 d_2)\left[\dfrac{32 - x_1}{d_1 d_2}\right]$ term. This is true of the other 3 "quartets" formed from the (d_1, d_2) pairs $(5,1)$, $(3,5)$ and $(3 \times 5, 1)$.
Accordingly the number of upper primes of twin prime pairs is given by:

$$T(5, 32, 2) = \sum^{*} \mu(d_1 d_2)\left[\frac{32 - x_1}{d_1 d_2}\right]$$

We want to prove this is true in general for any larger even positive integer N replacing 32 and p_θ replacing 5 where p_θ is the largest prime less than $\sqrt{N}$.

Chapter 3

Counting Theorem

3.1 Quartets and expanded Table 4

In constructing Table 4 we began the data entries with the two rows for the (d_1, d_2) pairs of (1,1) and (2,1). Thereafter whenever we introduced each of the succession of sieving primes we generated sets of four rows, one set for 3, three more for 5.

Definition 2. *We call these sets of four rows Quartets and note they all have the same form of,*

d_1	d_2
π_1	π_2
π_2	π_1
$2\pi_1$	π_2
$2\pi_2$	π_1

where π_1 *and* π_2 *are the products of integers chosen from* $\{3,5\} \cup \{1\}$ *with* $gcd(\pi_1, \pi_2) = 1$ *but excluding the* (d_1, d_2) *pairs of* $(1,1)$ *and* $(2,1)$ *which as noted above generated the first two rows.*

Specifically, in each quartet,

- the first row is formed from a unique selection of products of one or more elements of $\{1, 3, 5\}$ to form d_1 and d_2. ($d_1 \neq 1$.)
- the second row is formed by interchanging d_1 and d_2.
- the third row is formed from $2d_1$ and d_2.
- the fourth row is formed from $2d_2$ and d_1.

Definition 3. *For any combination of d_1 and d_2 each chosen from the product of elements of the set of primes and 1, that is from $\{2, 3, 5, \cdots, p_\theta\} \cup \{1\}$ with $gcd(d_1, d_2) = 1$ and $2 \nmid d_2$, we define the term x_1 to be the solution of the system of linear congruences,*

$$x \equiv 0(\text{mod } d_1) \qquad x \equiv 2(\text{mod } d_2)$$

Notation 3. *A quartet of rows in Table 4 can have four different values for x_1 and we will use the symbols $\alpha, \beta, \gamma, \delta$ for these four values. In particular, with "least non-negative" abbreviated to "lnn" we define,*

1. *α is the lnn solution of the system $x \equiv 0(\text{mod } d_1),\ x \equiv 2(\text{mod } d_2)$*
2. *β is the lnn solution of the system $x \equiv 0(\text{mod } d_2),\ x \equiv 2(\text{mod } d_1)$*
3. *γ is the lnn solution of the system $x \equiv 0(\text{mod } 2d_1),\ x \equiv 2(\text{mod } d_2)$*
4. *δ is the lnn solution of the system $x \equiv 0(\text{mod } 2d_2),\ x \equiv 2(\text{mod } d_1)$*

In all subsequent Table-4 like Tables, we will always put each quartet of rows in the above order. The four rows of each quartet in a table will be called the α, β, γ and δ rows and the sum of the $\pm 1's$ in each row will be referred to as $SR(\alpha)$, $SR(\beta)$, $SR(\gamma)$ and $SR(\delta)$ respectively. We now expand Table 4 with more sieving primes.

Definition 4. *When we introduce more sieving primes in the order $7, 11, 13, \ldots$ we need to construct an expanded Table 4. In general, when we introduce the p^{th} prime we add 3^{p-1} quartets of rows to Table 4 and we need to increase the numbered columns from 1 to 32 to 1 to a new N such that the p^{th} prime is the largest prime less than $\sqrt{N}$. We call all of this an expanded Table 4.*

Lemma 1. *The p^{th} sieving prime adds 3^{p-1} quartets to an expanded Table 4.*

Proof. We add the p^{th} sieving prime to an expanded Table 4. We designate it as p. The (d_1, d_2) pair of $(p, 1)$ gives the first quartet, the next $p-1$ quartets have $d_1 = p$ times each of the preceding sieving primes and $d_2 = 1$ and so on. We have a total of:

$$\begin{aligned}
&1 + \binom{p-1}{1} + \binom{p-1}{2} + \ldots \binom{p-1}{p-1} \\
&+ \binom{p-1}{1}\left[1 + \binom{p-2}{1} + \binom{p-2}{2} + \ldots + \binom{p-2}{p-2}\right] \\
&+ \binom{p-1}{2}\left[1 + \binom{p-3}{1} + \binom{p-3}{2} + \ldots + \binom{p-3}{p-3}\right] \\
&+ \ldots \\
&+ \binom{p-1}{p-1}[1] \\
&= 2^{p-1} + \binom{p-1}{1}2^{p-2} + \binom{p-1}{2}2^{p-3} + \ldots \binom{p-1}{p-1}2^0 \\
&= (2+1)^{p-1} = 3^{p-1} \text{ quartets}
\end{aligned}$$

$\square$

The headings of an expanded Table 4 conclude with columns for the integers 1 to N. Obviously we cannot print the expansions of Table 4 as we add more and more sieving primes from 7 to 11 to 13 and so on since the number of rows increases exponentially. But we can visualize and generalize its simple structure. Indeed, for counting purposes we only need to find the first and last terms in each row, the in-between values being known from the constructions $x_1,\ x_1 + d_1 d_2$ and so on described above.

- The first data row, due to $\mu(1)$, is simply N consecutive $+1's$. From now on we require N to be an even positive integer.
- The second data row begins with a -1 under 2 and then -1's under 2+2, 2+2+2, etc, the negative signs due to $\mu(2)$, giving a total of $\frac{N}{2}\ -1's$.
- Thereafter we have a series of quartets of rows as outlined above.

Our goal is to show the four sums of $\pm 1's$ in each quartet can be replaced by the sum of four terms of the form $\mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$ with each x_1 as defined above.
The sum of four values of SR in a general quartet can be easily calculated as we shall see but to get the grand total of all the $SR's$ we need the addition of the $SR's$ of,

$$4 \times (3^0 + 3^1 + 3^2 + \cdots + 3^{\theta-1})$$

rows where p_θ is prime number θ in the ascending size of primes. This is too long an addition for large θ and hence the shift to the form $\mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$
But we note that the value of SR for each row in an expanded Table 4 gives an actual count of the number of insertions and deletions and therefore the sum of the SRs is an actual count of the number of upper twin primes in the (p_θ, N) interval.
We have the following setup for the main counting theorem:

1. N is a positive even integer in the interval $(p_\theta, p_{\theta+1}^2)$ where,
2. p_θ is the largest prime less than $\sqrt{N}$ and $p_{\theta+1}$ is the next largest prime.
3. d_1 is the product of one or more elements of the set $\{2, 3, \ldots, p_\theta\} \cup \{1\}$, where $2, 3, \ldots p_\theta$ are consecutive primes,
4. d_2 is the product of one or more elements of the set $\{3, \ldots, p_\theta\} \cup \{1\}$ subject to the conditions that $gcd(d_1, d_2) = 1$ and also $gcd(d_2, 2) = 1$,

Our goal is to show $\sum \mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$, with the sum limits to be defined, is equal to the sum of the SRs, therefore being a count of all the twin prime pairs in this interval and therefore we have $T(p_\theta, N, 2) = \sum \mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$.

3.2 Two lemmas and two corollaries

Here are four results we will need for what follows. We will deal separately with the cases where $d_1 = 1$ or $d_2 = 1$ so in what follows we always require $d_1 \neq 1$ and $d_2 \neq 1$.

Lemma 2.
For any quartet in an expanded Table 4 with $d_1 > 1$ and $d_2 > 1$ and both a product of odd primes, we have,

$$\alpha + \beta = 2 + d_1 d_2.$$

So one of α and β is odd and the other is even.

Proof.
By definition,
α is the least non-negative solution of the system $x \equiv 0(\bmod\ d_1)$ and $x \equiv 2(\bmod\ d_2)$,
β is the least non-negative solution of the system $x \equiv 0(\bmod\ d_2)$ and $x \equiv 2(\bmod\ d_1)$.
So from the definitions of α and β we have for some $a, b, c, e \in \mathbb{Z}^+$,

$$\begin{aligned} \alpha &= ad_1 = 2 + bd_2 \\ \beta &= cd_2 = 2 + ed_1 \\ \Rightarrow \alpha + \beta &= 2 + bd_2 + cd_2 \equiv 2(\bmod\ d_2) \\ \alpha + \beta &= ad_1 + 2 + ed_1 \equiv 2(\bmod\ d_1) \end{aligned}$$

Now since each prime number in both d_1 and d_2 divides $\alpha + \beta - 2$ then,

$$d_1 d_2 | \alpha + \beta - 2 \Rightarrow \alpha + \beta \equiv 2(\bmod\ d_1 d_2) \Rightarrow \alpha + \beta = 2 + k d_1 d_2,\ \ k \in \mathbb{Z}^+$$

Now $\alpha = ad_1 \Rightarrow \alpha > 2$ since $d_1 \geq 3$ and $\beta = cd_2 \Rightarrow \beta > 2$ so

$$\alpha + \beta > 4$$

Note, by the Chinese Remainder Theorem that all solutions of linear congruences such as,

$$x \equiv 0(\bmod\ d_1) \text{ and } x \equiv 2(\bmod\ d_2)$$

are equal congruent modulo $d_1 d_2$ and therefore α as the least non-negative solution is less than $d_1 d_2$. The same applies to β.
Now $\alpha < d_1 d_2$ and $\beta < d_1 d_2$ makes $\alpha + \beta < 2d_1 d_2$. Hence

$$4 < \alpha + \beta < 2d_1 d_2$$

But $\alpha + \beta = 2 + kd_1 d_2$, hence k cannot be 0 or greater than 1. Accordingly,

$$\alpha + \beta = 2 + d_1 d_2$$

$\square$

Corollary 3.
One of α and β is greater than $\dfrac{2+d_1d_2}{2}$ and the other is less than $\dfrac{2+d_1d_2}{2}$.
Accordingly, if $\alpha > \beta$ then $\alpha > \dfrac{2+d_1d_2}{2} \Rightarrow 2\alpha - 2 > d_1d_2$ and if $\alpha < \beta$ then $2\alpha - 2 < d_1d_2$.

Lemma 4.
If α is odd then $\gamma = \alpha + d_1d_2$
If α is even then $\gamma = \alpha$.
The same relationship applies to β and δ.

Proof. Suppose α is odd. We have,
$\alpha \equiv 0 (\bmod\ d_1) \Rightarrow \alpha = ad_1, a \in \mathbb{Z}^+$ and we must have a odd since α is.
Also, $\alpha \equiv 2 (\bmod\ d_2) \Rightarrow \alpha = 2 + kd_2$.
We want to show $\gamma = \alpha + d_1d_2$ where,

$$\gamma \equiv 0 (\bmod\ 2d_1),\ \gamma \equiv 2 (\bmod\ d_2)$$

Now,

$$\alpha + d_1d_2 = 2 + kd_2 + d_1d_2 \equiv 2 (\bmod\ d_2)$$

as is γ. And,

$$\alpha + d_1d_2 = ad_1 + d_1d_2 = (a + d_2)d_1$$

Now a is odd so $a + d_2$ is even, say $a + d_2 = 2j$. Then,

$$\alpha + d_1d_2 = 2jd_1 \equiv 0 (\bmod\ 2d_1)$$

as is γ. Hence,

$$\gamma = \alpha + d_1d_2.$$

****

Suppose α is even. Again note $\gamma \equiv 0 (\bmod\ 2d_1)$ and $\gamma \equiv 2 (\bmod\ d_2)$.
Now $\alpha = ad_1$ makes a even. So we can write $\alpha = 2\bar{a}d_1 \equiv 0 (\bmod\ 2d_1)$. But together with $\alpha \equiv 2 (\bmod\ d_2)$, this is the definition of γ. □

Corollary 5.

$$\gamma + \delta = 2 + 2d_1d_2$$

Proof. From Lemma 2 one of α, β is odd and the other even.
So either $\gamma = \alpha$ and $\delta = \beta + d_1d_2$ or vice versa. Hence, by Lemma 8,

$$\gamma + \delta = \alpha + \beta + d_1d_2 = 2 + 2d_1d_2$$

□

Using the terminology as above we will prove a series of lemmas which together give us the proof of the following theorem.

3.3 Main Counting Theorem

Theorem 6.
In the primod-p_θ number system, the number of primods less than a non-negative, even integer, N, with no p-digits equal to 0 or $2(\text{mod } p)$ is given by:

$$T(p_\theta, N, 2) = \sum^{*} \mu(d_1 d_2) \left[\frac{N - x_1}{d_1 d_2} \right]$$

where $\sum^{*}$ is a sum where,

(a) p_θ *is the largest prime less than* $\sqrt{N}$,

(b) d_1 *is the product of one or more elements of the set* $\{2, 3, \ldots, p_\theta\} \cup \{1\}$ *, where* $2, 3, \ldots p_\theta$ *are consecutive primes,*

(c) d_2 *is the product of one or more elements of the set* $\{3, \ldots, p_\theta\} \cup \{1\}$ *so that* $gcd(d_2, 2) = 1$,

(d) $gcd(d_1, d_2) = 1$ *or* $2 \nmid d_2$,

(e) the sum is over all possible values of d_1 *and* d_2,

(f) μ *is the Möbius function and* $[x]$ *the greatest integer function,*

(g) x_1 *is the least non negative solution of the system of equations*

$$x \equiv 0(\text{mod } d_1), \; x \equiv 2(\text{mod } d_2)$$

Equivalently the number of upper primes of a twin prime pair between p_θ *and* N *is given by* $T(p_\theta, N, 2)$ *as defined above.*

Proof.
Without actually showing it, since its size grows exponentially, we visualize expanded Table 4s constructed by introducing successive primes into such Table 4s beginning with 7, then 11, and so on up to p_θ where p_θ is the largest prime less than $\sqrt{N}$, with $N : p_\theta < N < p_{\theta+1}^2$. In each expanded Table 4 we need show only the first and last terms in a row since the first term is that row's x_1 and all the other terms are $x_1 + d_1 d_2, x_1 + 2d_1 d_2, \ldots$ for insertions or $x_1 + 2d_1 d_2, x_1 + 4d_1 d_2, \ldots$ for deletions, stopping if such an insertion/deletion exceeds N and giving the last term for that row.
It is simplist to think of N as $p_\theta^2 < N < p_{\theta+1}^2$ so that N is readjusted as new $p_\theta' s$ are introduced. First we show the SR sums and the $\mu(d_1 d_2) \left[\frac{N - x_1}{d_1 d_2} \right]$ values are the same for the first two data rows in an expanded Table 4.

3.3.1 First two rows

Lemma 7. *In any expanded Table 4 the SR sum of the first row is N, the SR sum of the second row is $-\dfrac{N}{2}$ and the corresponding values of $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$ are also N and $-\dfrac{N}{2}$ respectively.*

Proof. As in Table 4, there will be a "1" placed under each of the integers 1 to N making the sum of the first row $SR = N$. In this row, $d_1 = d_2 = 1$ so $x_1 = 0$ since it is the least non-negative solution of $x \equiv 0 (\text{mod } 1)$ and $x \equiv 2 (\text{mod } 1)$ so that $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right] = N$ also.
For the second row there will be a "-1" placed under each even integer in the range 1 to N and since N is even, this $SR = -\dfrac{N}{2}$. In this row, $d_1 = 2$ and $d_2 = 1$ so that $x_1 = 0$ again and $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right] = -\dfrac{N}{2}$ also. □

3.3.2 Quartets with $\pi_2 = 1$ and $\pi_1 > 1$.

Lemma 8. *With reference to the general Definition 2 on page 16 of a quartet we put $\pi_1 = \pi$ and $\pi_2 = 1$. For π a product of odd primes chosen from $\{3, 5, 7, \ldots, p_\theta\}$ we have the unique quartet,*

d_1	d_2	d_1d_2	$\mu(d_1d_2)$	x_1
π	1	π	1	0
1	π	π	1	2
2π	1	2π	-1	0
2	π	2π	-1	2

Table 5: Quartets with $\pi_2 = 1$

where, since the choice does not matter, we choose $\mu(\pi) = 1$ and $\mu(2\pi) = -1$. Then the sum of the respective four values of SR in an expanded Table 4 equal the corresponding sum of the four values of $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$

Proof. Let π be a product of odd primes chosen from $\{3, 5, 7, \ldots, p_\theta\}$.
The four values for x_1 in the final column of Table 5 are the solutions of the following systems of linear congruences and we again label them $\alpha, \beta, \gamma, \delta$ to distinguish them.

$$x \equiv 0 (\text{mod } \pi),\ x \equiv 2 (\text{mod } 1) \Rightarrow \alpha = 0$$
$$x \equiv 0 (\text{mod } 1),\ x \equiv 2 (\text{mod } \pi) \Rightarrow \beta = 2$$
$$x \equiv 0 (\text{mod } 2\pi),\ x \equiv 2 (\text{mod } 1) \Rightarrow \gamma = 0$$
$$x \equiv 0 (\text{mod } 2),\ x \equiv 2 (\text{mod } \pi) \Rightarrow \delta = 2$$

There are 8 cases to consider. Note,

1. Each choice for N must be even so N cannot equal π.
2. The cases $N = k\pi + l, k$ even, $l < \pi$ are included in $N = 2k\pi + l$ and the cases $N = k'\pi + l, k'$ odd in $N = 2k\pi + \pi + l, k' = 2k + 1$.
3. We have $k, k' \in \mathbb{Z}^+$.
4. We have $\pi \geq 3$.

Case 1: $2 < N < \pi$
Case 2: $N = \pi + 1$
Case 3: $N = \pi + l, 3 \leq l < \pi, l$ odd (note $\pi + 2$ is an odd number).
Case 4: $\pi | N$ or $N = 2k\pi,\ k \in \mathbb{Z}^+$
Case 5: $N = 2k\pi + 2$ (note $N = 2k\pi + 1$ is an odd number).
Case 6: $N = 2k\pi + l,\ 2 < l < \pi,\ l$ even
Case 7: $N = 2k\pi + \pi + 1$
Case 8: $N = 2k\pi + \pi + l,\ 2 < l < \pi,\ l$ odd

We consider each of the 8 cases in turn.
Note each d_1, d_2 choice is unique and connects the four SR values with the four greatest integer values. In each quartet we align the four SR values with the four greatest integer values by arbitrarily making the insertions into the α and β rows all +1's and the γ and δ rows all insertions of $-1's$. Then the $\mu(d_1d_2)$ for the first two rows we put $+1's$ and for the last two rows we put $-1'$ since we are dealing with an extra prime, namely 2, in $\mu(2d_1d_2)$. Clearly we could reverse all this but the overall conclusion as to whether the sums of the four SR rows and the four greatest integer terms are equal or not would be the same.
We fill in each row of our unique quartet of an expanded Table 4 beginning with a ± 1 at the position of its x_1 value and then further $\pm 1's$ at the positions of successive additions of d_1d_2 for the α and β rows and at successive additions of $2d_1d_2$ for the γ and δ rows.
Each of the 8 tables below shows the position of the first entry and the last entry of the relevant row in an expanded Table 4 at which they and the inbetween positions we would put the +1 or -1, from which we calculate the SR values.
Note each last entry must be $\leq N$.
In the 8 cases calculated in Tables 6 to 13, the four x_1 values are $\alpha = 0, \beta = 2$, $\gamma = 0, \delta = 2$.
The SR values are straightforward, being the number of entries between the first and last entry on the respective row of an expanded Table 4.
The $\mu(d_1d_2)\left[\dfrac{N - x_1}{d_1d_2}\right]$ values are calculated for each case. For convenience we choose $\mu(d_1d_2) = 1$ and $\mu(2d_1d_2) = -1$.

Case 1: $2 < N < \pi$
We have, noting $\pi \geq 3$,

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{<\pi}{\pi}\right] = 0 \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{<(\pi-2)}{\pi}\right] = 0$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{<\pi}{2\pi}\right] = 0 \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{<(\pi-2)}{2\pi}\right] = 0$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	0	0	0	0
β	1	π	π	2	2	2	1	0
γ	2π	1	2π	0	0	0	0	0
δ	2	π	2π	2	2	2	−1	0
						Total	0	0

Table 6 - Case 1: $2 < N < \pi$

We calculate the SR sums as follows. The possible entries on the α row where $\alpha = 0$ are at the positions $0,\ 0+\pi,\ 0+2\pi,\ldots$ But with $N < \pi$ we only have 0 as the first and last entry position before N. Thus $SR(\alpha) = 0$. The possible entries on the β row where $\beta = 2$ are at the positions $2,\ 2+\pi,\ 2+2\pi,\ldots$ but with $N < \pi$ we only have 2 as the first and last entry position on the β row and thus $SR(\beta) = 1$. Note +1 and not −1 since we are arbitrarily choosing the α and β rows are additions.
The possible entries on the γ row where $\gamma = 0$ are at positions $0,\ 0+2d_1d_2,\ 0+4d_1d_2\ldots$ but with $N < \pi$ we only have 0 as the first and last entry position, hence $SR(\gamma) = 0$. Finally the possible entry positions on the δ row are at $2,\ 2+2d_1d_2, 2+4d_1d_1,\ \ldots$ but again with $N < \pi$ we only have 2 as the first and last insertion position, making $SR(\delta) = -1$. Note it is −1 and not +1 since if the α and β rows are for additions then, with 2 as an extra prime multiplying d_1d_2, the γ and δ rows are for deletions.
Similar arguments apply to all the other Cases and Cases 2 and 3 are presented without further comment.

Case 2: $N = \pi + 1$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{\pi+1}{\pi}\right] = 1 \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{\pi+1-2}{\pi}\right] = 0$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{\pi+1}{2\pi}\right] = 0 \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{\pi+1-2}{2\pi}\right] = 0$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	π	1	1
β	1	π	π	2	2	2	1	0
γ	2π	1	2π	0	0	0	0	0
δ	2	π	2π	2	2	2	-1	0
						Total	1	1

Table 7 - Case 2: $N = \pi + 1$

Case 3: $N = \pi + l, 3 \le l < \pi$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{\pi+l}{\pi}\right] = 1 \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{\pi+l-2}{\pi}\right] = 1$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{\pi+l}{2\pi}\right] = 0 \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{\pi-2+l}{2\pi}\right] = 0$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	π	1	1
β	1	π	π	2	2	$\pi+2$	2	1
γ	2π	1	2π	0	0	0	0	0
2δ	2	π	2π	2	2	2	-1	0
						Total	2	2

Table 8 - Case 3: $N = \pi + l,\ 3 \le l < \pi$

Here are some comments on Case 4, the arguments for Cases 5 to 8 are similar. The possible entry positions on the α row where $\alpha = 0$ are $0,\ \pi,\ 2\pi, \ldots, 2k\pi = N$ so $SR(\alpha) = 2k\pi$. The possible entry positions on the β row where $\beta = 2$ are $2,\ \pi+2,\ 2\pi+2, \ldots, (2k-1)\pi+2,\ 2k\pi+2, \ldots$. Noting $(2k-1)\pi+2$ is the last entry less than $N = 2k\pi$ then $SR(\beta) = 2k$. The possible entry positions on the γ row where $\gamma = 0$ are $0,\ 2\pi,\ 4\pi,\ \ldots k(2\pi) = N$ so $SR(\gamma) = -k$. Finally the possible entry positions on the δ row where $\delta = 2$ are $2,\ 2\pi+2,\ 4\pi+2,\ \ldots, (k-1)2\pi+2$ so $SR(\delta) = -k$. Note 2π is much larger than 2 so $(k-1)2\pi + 2 < N = 2k\pi$ but $2k\pi + 2 > N = 2k\pi$.

Case 4: $\pi|N$ or $N = 2k\pi,\ k \in \mathbb{Z}^+$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2k\pi}{\pi}\right] = 2k \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{2k\pi-2}{\pi}\right] = 2k-1$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2k\pi}{2\pi}\right] = k \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{2k\pi-2}{2\pi}\right] = k-1.$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	$2k\pi$	$2k$	$2k$
β	1	π	π	2	2	$2+(2k-1)\pi$	$2k$	$2k-1$
γ	2π	1	2π	0	2π	$2k\pi$	$-k$	$-k$
δ	2	π	2π	2	2	$2+(k-1)2\pi$	$-k$	$-(k-1)$
						Total	$2k$	$2k$

Table 9 - Case 4: $N = 2k\pi,\ k \in \mathbb{Z}^+$

Case 5: $N = 2k\pi + 2$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2k\pi+2}{\pi}\right] = 2k \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{2k\pi}{\pi}\right] = 2k$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2k\pi+2}{2\pi}\right] = k \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{2k\pi}{2\pi}\right] = k$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	$2k\pi$	$2k$	$2k$
β	1	π	π	2	2	$2+2k\pi$	$(2k+1)$	$2k$
γ	2π	1	2π	0	2π	$2k\pi$	$-k$	$-k$
δ	2	π	2π	2	2	$2+2k\pi$	$-(k+1)$	$-k$
						Total	$2k$	$2k$

Table 10 - Case 5: $N = 2k\pi + 2$

Case 6: $N = 2k\pi + l,\ 2 < l < \pi$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2k\pi+l}{\pi}\right] = 2k \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{2k\pi+l-2}{\pi}\right] = 2k$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2k\pi+l}{2\pi}\right] = k \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{2k\pi+l-2}{2\pi}\right] = k$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	$2k\pi$	$2k$	$2k$
β	1	π	π	2	2	$2+2k\pi$	$(2k+1)$	$2k$
γ	2π	1	2π	0	2π	$2k\pi$	$-k$	$-k$
δ	2	π	2π	2	2	$2+2k\pi$	$-(k+1)$	$-k$
						Total	$2k$	$2k$

Table 11 - Case 6: $N = 2k\pi + l,\ 2 < l < \pi$

Case 7: $N = 2k\pi + \pi + 1$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2k\pi+\pi+1}{\pi}\right] = 2k+1 \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{2k\pi+\pi+1-2}{\pi}\right] = 2k$$

$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2k\pi+\pi+1}{2\pi}\right] = k \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{2k\pi+\pi+1-2}{2\pi}\right] = k$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	$(2k+1)\pi$	$(2k+1)$	$(2k+1)$
β	1	π	π	2	2	$2+2k\pi$	$(2k+1)$	$2k$
γ	2π	1	2π	0	2π	$2k\pi$	$-k$	$-k$
δ	2	π	2π	2	2	$2+2k\pi$	$-(k+1)$	$-k$
						Total	$(2k+1)$	$(2k+1)$

Table 12 - Case 7: $N = 2k\pi + \pi + 1$

Case 8: $N = 2k\pi + \pi + l,\ 2 < l < \pi$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2k\pi+\pi+l}{\pi}\right] = 2k+1 \qquad \left[\frac{N-\beta}{d_1d_2}\right] = \left[\frac{2k\pi+\pi+l-2}{\pi}\right] = 2k+1$$

$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2k\pi+\pi+l}{2\pi}\right] = k \qquad \left[\frac{N-\delta}{2d_1d_2}\right] = \left[\frac{2k\pi+\pi+l-2}{2\pi}\right] = k$$

Row	d_1	d_2	d_1d_2	x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$
α	π	1	π	0	π	$(2k+1)\pi$	$(2k+1)$	$(2k+1)$
β	1	π	π	2	2	$2+(2k+1)\pi$	$(2k+2)$	$(2k+1)$
γ	2π	1	2π	0	2π	$2k\pi$	$-k$	$-k$
δ	2	π	2π	2	2	$2+2k\pi$	$-(k+1)$	$-k$
						Total	$(2k+2)$	$(2k+2)$

Table 13 - Case 8: $N = 2k\pi + \pi + l,\ 2 < l < \pi$

The Lemma is proved. In all eight cases of the specified quartets, the sum of the respective four values of SR in the expanded Table 4 equal the corresponding sum of the four values of $\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$. □

All the other quartets in an expanded Table 4 have d_1 and d_2 both greater than 1. In general we choose from the primes $\{3, 5, \ldots p_\theta\}$, a product of odd primes for d_1 which we label $d_1 = \prod^* p_i$ and a different product of odd primes for d_2 which we label $d_2 = \prod^* q_j$ and note no q_j is the same as any p_i or $gcd(d_1, d_2) = 1$. Then the general quartet will be formed thus,

d_1	d_2	d_1d_2
$\prod^* p_i$	$\prod^* q_j$	$\prod^* p_i \prod^* q_j$
$\prod^* q_j$	$\prod^* p_i$	$\prod^* p_i \prod^* q_j$
$2\prod^* p_i$	$\prod^* q_j$	$2\prod^* p_i \prod^* q_j$
$2\prod^* q_j$	$\prod^* p_i$	$2\prod^* p_i \prod^* q_j$

Table 14: General Quartet

We again label the four corresponding values of x_1 as,
α: the least non negative solution of $x \equiv 0(\text{mod } \prod^* p_i)$ and $x \equiv 2(\text{mod } \prod^* q_j)$
β: the least non negative solution of $x \equiv 0(\text{mod } \prod^* q_j)$ and $x \equiv 2(\text{mod } \prod^* p_i)$
γ: the least non negative solution of $x \equiv 0(\text{mod } 2\prod^* p_i)$ and $x \equiv 2(\text{mod } \prod^* q_j)$
δ: the least non negative solution of $x \equiv 0(\text{mod } 2\prod^* q_j)$ and $x \equiv 2(\text{mod } \prod^* p_i)$

Having dealt with the first two data lines and all quartets with $d_1 = 1$ or $d_2 = 1$ in expanded Table 4s, we can analyze all the other quartets in an expanded Table 4 by noting either $\alpha = N$, $\alpha > N$ or $\alpha < N$. We note that N and α are independent of one another and that the values of β, γ and δ are derived from the values of d_1 and d_2 that give α, so the four of them are related as shown in Lemmas 2 and 4 and Corollaries 3 and 5. Note also that both α and β are less than their respective d_1d_2 and therefore so is $\alpha - \beta$ if $\alpha > \beta$ or $\beta - \alpha$ if $\beta > \alpha$.

3.3.3 Quartets with $\alpha = N$.

Lemma 9. *For those quartets with $\alpha = N$ the sum of the respective four values of SR in an expanded Table 7 equal the sum of the corresponding four values of* $\mu(d_1d_2)\left[\dfrac{N - x_1}{d_1d_2}\right]$

Proof. Since N is even we have $\alpha = \gamma = N$. There are two cases,
Case 1: $\alpha = N, \beta < N$
Case 2 $\alpha = N, \beta > N$
Note since $\alpha + \beta = 2 + d_1d_2$ and d_1d_2 is odd, we cannot have $\alpha = \beta$.

Case 1: $\alpha = N, \beta < N$
By Lemma 4, $\gamma = \alpha = N$ since N and therefore α are even. And we have $\delta = \beta + d_1d_2$ since β must be odd.
We have $SR(\alpha) = 1$ since $\alpha = N$ is counted but $\alpha + d_1d_2 > N$ is not.
Similarly $SR(\gamma) = -1$.
We are given $\beta < N$ but both are less than d_1d_2 since α is, so $N - \beta < d_1d_2$. Then $\beta + d_1d_2 > N$ making $SR(\beta) = 1$.

By the same argument $\delta = \beta + d_1 d_2 > N$ so $SR(\delta) = 0$. Finally,

$$\left[\frac{N-\alpha}{d_1 d_2}\right] = 0 \qquad \left[\frac{N-\gamma}{2d_1 d_2}\right] = 0$$

$$N - \beta > 0 \text{ and } N - \beta < d_1 d_2 \Rightarrow 0 < N - \beta < d_1 d_2 \Rightarrow \left[\frac{N-\beta}{d_1 d_2}\right] = 0$$

$$-d_1 d_2 < N - \beta - d_1 d_2 < 0 \Rightarrow \left[\frac{N-\delta}{2d_1 d_2}\right] = -1$$

giving Table 15 for Case 1 below. The Total given in the final column of each Table of Case 1 and Case 2 is the sum of the entries in the last two columns, that is of the four greatest integer calculations. This will apply to all future cases.

x_1	First entry	Last entry	SR	$\mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$	$\mu(2d_1 d_2)\left[\frac{N-x_1}{2d_1 d_2}\right]$
α	α	α	1	0	
β	β	β	1	0	
γ	γ	γ	−1		0
δ	0	0	0		1
Total			1		1

Table 15 - Case 1: $\alpha = N, \beta \leq N$

Case 2: $\alpha = N,\ \beta > N$
We have $\gamma = \alpha = N$ since N and therefore α are both even. And we have $\delta = \beta + d_1 d_2$ since β must be odd. Then,

$$\left[\frac{N-\alpha}{d_1 d_2}\right] = 0 \qquad \left[\frac{N-\gamma}{2d_1 d_2}\right] = 0$$

$$N - \beta < 0 \text{ and } \beta - N < d_1 d_2 \Rightarrow -d_1 d_2 < N - \beta < 0 \Rightarrow \left[\frac{N-\beta}{d_1 d_2}\right] = -1$$

$$-2d_1 d_2 < N - \beta - d_1 d_2 < -d_1 d_2 \Rightarrow \left[\frac{N-\delta}{2d_1 d_2}\right] = -1$$

giving the Table for Case 2 below.

x_1	First entry	Last entry	SR	$\mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$	$\mu(2d_1 d_2)\left[\frac{N-x_1}{2d_1 d_2}\right]$
α	α	α	1	0	
β	0	0	0	−1	
γ	γ	γ	−1		0
δ	δ	0	0		1
Total			0		0

Table 16 - Case 2: $\alpha = N,\ \beta > N$

The Lemma is proved. In both cases the sum of the respective four SR of such quartets in an expanded Table 4 equal the corresponding sum of the four values of $\mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$. □

3.3.4 Quartets with $\alpha > N$.

Lemma 10. *For those quartets with $\alpha > N$ the sum of the respective four values of SR in an expanded Table4 equal the sum of the corresponding four values of* $\mu(d_1 d_2)\left[\dfrac{N-x_1}{d_1 d_2}\right]$

Proof.
In any quartet there are four possibilities or cases for $\alpha > N$.

Case 1: $\alpha > N,\ \beta > N,\ \alpha\ even, \beta\ odd$
Case 2: $\alpha > N,\ \beta < N,\ \alpha\ even, \beta\ odd$
Case 3: $\alpha > N,\ \beta > N,\ \alpha\ odd, \beta\ even$
Case 4: $\alpha > N,\ \beta < N,\ \alpha\ odd, \beta\ even$

Note $\alpha < d_1 d_2$ and $\alpha > N$ means we also have $N < d_1 d_2$.

Case 1: $\alpha > N,\ \beta > N,\ \alpha\ even, \beta\ odd$
We have $\gamma = \alpha$ and $\delta = \beta + d_1 d_2$ so all four of $\alpha, \beta, \gamma, \delta$ are greater then N and there will be no entries in the rows of an expanded Table 4 corresponding to this quartet so that the four SR sums are all zero.
Since $\alpha > N$ then both are less than $d_1 d_2$ and so is their difference and we have $-d_1 d_2 < N - \alpha < 0$ so that $\left[\dfrac{N-\alpha}{d_1 d_2}\right] = -1$.
Similarly $-d_1 d_2 < N - \beta < 0 \Rightarrow \left[\dfrac{N-\beta}{d_1 d_2}\right] = -1$.
Since $\gamma = \alpha$ then $\left[\dfrac{N-\gamma}{2d_1 d_2}\right] = -1$.
Finally, $-d_1 d_2 < N - \beta < 0 \Rightarrow -2d_1 d_2 < N - \delta < -d_1 d_2 \Rightarrow \left[\dfrac{N-\delta}{2d_1 d_2}\right] = -1$.
Hence we have,

x_1	First entry	Last entry	SR	$\mu(d_1 d_2)\left[\frac{N-x_1}{d_1 d_2}\right]$	$\mu(2d_1 d_2)\left[\frac{N-x_1}{2d_1 d_2}\right]$
α	0	0	0	-1	
β	0	0	0	-1	
γ	0	0	0		1
δ	0	0	0		1
Total			0		0

Table 17 - Case 1: $\alpha > N,\ \beta < N,\ \alpha\ even, \beta\ odd$

Case 2: $\alpha > N,\ \beta < N,\ \alpha\ even, \beta\ odd$
We have $\gamma = \alpha$ so α, γ are greater then N and the values in the α and γ rows are the same as in Case 1.
Now $\alpha < d_1 d_2$ and $\beta < d_1 d_2$ so $\alpha - \beta < d_1 d_2 \Rightarrow \alpha < \beta + d_1 d_2 = \delta$ so there are no entries on the δ row.
However, $\beta < N$ means there is an entry in the β row on an expanded Table 4 but no entry in any of the other three rows, giving the four SR values of 0,1,0,0 respectively.
Since $\alpha > N$ then both are less than $d_1 d_2$ and so is their difference and we have

$-d_1d_2 < N - \alpha < 0 \Rightarrow -1 < \dfrac{N-\alpha}{d_1d_2} < 0$ so that $\left[\dfrac{N-\alpha}{d_1d_2}\right] = -1$.

Since $\gamma = \alpha$ then $\left[\dfrac{N-\gamma}{2d_1d_2}\right] = -1$.

Since both N and β are less than d_1d_2 then $0 < N - \beta < d_1d_2$ so $\left[\dfrac{N-\beta}{d_1d_2}\right] = 0$.

Finally $0 < N - \beta < d_1d_2 \Rightarrow -d_1d_2 < N - \delta < 0$ so $\left[\dfrac{N-\delta}{2d_1d_2}\right] = -1$. Hence we have,

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	0	0	0	-1	
β	β	β	1	0	
γ	0	0	0		1
δ	0	0	0		1
Total			1		1

Table 18 - Case 2: $\alpha > N,\ \beta < N,\ \alpha\ even, \beta\ odd$

Case 3: $\alpha > N,\ \beta > N,\ \alpha\ odd, \beta\ even$
Since $\alpha > N, \beta > N$ then $SR(\alpha) = SR(\beta) = 0$.
Since $\gamma = \alpha + d_1d_2$ then $\gamma > N$ so $SR(\gamma) = 0$.
Since $\delta = \beta$ then $\delta > N$ so $SR(\delta) = 0$.
Since $\alpha > N$ then both are less than d_1d_2 and $-d_1d_2 < N - \alpha < 0$
$\Rightarrow -1 < \dfrac{N-\alpha}{d_1d_2} < 0$ making $\left[\dfrac{N-\alpha}{d_1d_2}\right] = -1$.

Similarly, $-d_1d_2 < N - \beta < 0 \Rightarrow -1 < \dfrac{N-\beta}{d_1d_2} < 0$ making $\left[\dfrac{N-\beta}{d_1d_2}\right] = -1$.

Since $\gamma = \alpha + d_1d_2$ then $-d_1d_2 < N - \alpha < 0 \Rightarrow -2d_1d_2 < N - \gamma < -d_1d_2 \Rightarrow \left[\dfrac{N-\gamma}{2d_1d_2}\right] = -1$.

Since $\delta = \beta$ we have $-d_1d_2 < N - \delta < 0 \Rightarrow -\dfrac{1}{2} < \dfrac{N-\delta}{2d_1d_2} < 0$ making $\left[\dfrac{N-\delta}{2d_1d_2}\right] = -1$.

We have,

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	0	0	0	-1	
β	0	0	0	-1	
γ	0	0	0		1
δ	0	0	0		1
Total			0		0

Table 19 - Case 3: $\alpha > N,\ \beta > N,\ \alpha\ odd, \beta\ even$

Case 4: $\alpha > N,\ \beta < N,\ \alpha\ odd, \beta\ even$
We have $\gamma = \alpha + d_1d_2$ and $\delta = \beta$ so α, γ are greater then N with each $SR = 0$ while $\beta < N \Rightarrow SR(\beta) = 1$ and $SR(\delta) = -1$.

Since $\alpha > N$, then as in Case 3, $\left[\dfrac{N-\alpha}{d_1d_2}\right] = -1$.

Since $\gamma = \alpha + d_1 d_2$ then as for Case 3, $\left[\dfrac{N-\gamma}{2d_1d_2}\right] = -1$.
Since $\beta < N$ then $N - \beta$ is a positive number less than d_1d_2 so $0 < N - \beta < d_1d_2 \Rightarrow$ $\left[\dfrac{N-\beta}{d_1d_2}\right] = 0$.
Since $\delta = \beta$ then $0 < N - \delta < d_1d_2 \Rightarrow \left[\dfrac{N-\delta}{2d_1d_2}\right] = 0$ also.

x_1	First terms	Last term	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	0	0	0	−1	
β	β	β	1	0	
γ	0	0	0		1
δ	δ	δ	−1		0
Total			0		0

Table20 - Case 4: $\alpha > N,\ \beta < N,\ \alpha\ odd, \beta\ even$

The Lemma is proved. In all quartets in an expanded Table 4 where $\alpha > N$, the four SR sums add to the same value as the two $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$ values plus the two $\mu(2d_1d_2)\left[\dfrac{N-x_1}{2d_1d_2}\right]$ values. □

3.3.5 Quartets with $\alpha < N$.

Lemma 11. *For those quartets with $\alpha < N$ the sum of the respective four values of SR in an expanded Table 4 equal the sum of the corresponding four values of* $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$

Proof. There are 19 cases to consider, most with an A and B option.
Note each choice for N must be even.
Note the cases $N = kd_1d_2 + l, k$ even are included in $N = 2kd_1d_2 + l$ and the cases $N = k'd_1d_2 + l, k'$ odd in $N = 2kd_1d_2 + d_1d_2 + l$ where $k' = 2k + 1$. In Cases 10 to 19 $k \geq 1,\ k \in \mathbb{Z}$

Case 1: $N - \alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ even so l is even.
Case 2: $N - \alpha = l,\ l < d_1d_2, \alpha < \beta, \alpha$ even so l is even.
Case 3: $N - \alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd so l is odd.
Case 4: $N - \alpha = l,\ l < d_1d_2, \alpha < \beta, \alpha$ odd so l is odd.
Case 5: $N - \alpha = d_1d_2$, so α is odd.
Case 6: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha > \beta, \alpha$ even so l is odd.
Case 7: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha < \beta, \alpha$ even so l is odd.
Case 8: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd so l is even.
Case 9: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha < \beta, \alpha$ odd so l is even.
Case 10: $N - \alpha = 2kd_1d_2,\ \alpha$ even, $\alpha > \beta$

Case 11: $N - \alpha = 2kd_1d_2$, α even, $\beta > \alpha$
Case 12: $N - \alpha = 2kd_1d_2 + l$, $l < d_1d_2, \alpha, l$ even, $\alpha > \beta$
Case 13: $N - \alpha = 2kd_1d_2 + l$, $l < d_1d_2, \alpha, l$ even, $\beta > \alpha$
Case 14: $N - \alpha = 2kd_1d_2 + l$, $l < d_1d_2$, α, l odd, $\alpha > \beta$
Case 15: $N - \alpha = 2kd_1d_2 + l$, $l < d_1d_2$, α, l odd, $\beta > \alpha$
Case 16: $N - \alpha = 2kd_1d_2 + d_1d_2 + l$, $l < d_1d_2, l$, odd, α even, $\alpha > \beta$
Case 17: $N - \alpha = 2kd_1d_2 + d_1d_2 + l$, $l < d_1d_2, l$ odd, α even, $\beta > \alpha$
Case 18: $N - \alpha = 2kd_1d_2 + d_1d_2 + l$, $l < d_1d_2, l$ even, α odd, $\alpha > \beta$
Case 19: $N - \alpha = 2kd_1d_2 + d_1d_2 + l$, $l < d_1d_2, l$ even, α odd, $\beta > \alpha$

Note 1. *Note $\alpha > \beta$ generates a chain of inequalities such as $\beta < \alpha < \beta + d_1d_2 < \alpha + d_1d_2 < \beta + 2d_1d_2, \cdots$ since $\alpha < d_1d_2$ and $\beta < d_1d_2$ makes $\alpha - \beta < d_1d_2$ or $\alpha < \beta + d_1d_2$ and so on.*
Note also when counting for $SR(\gamma)$ and $SR(\delta)$ the counted terms are of the form $\gamma,\ \gamma + 2d_1d_2,\ \gamma + 4d_1d_2, \ldots$ and similarly for the δ row, thus terms of the form $\gamma + d_1d_2$ or $\delta + d_1d_2$ are not counted in the SRs.
However, for example, the chain $\gamma + d_1d_2 < N < \gamma + 2d_1d_2$ does make $\left[\dfrac{N-\gamma}{2d_1d_2}\right] = 0$.

Case 1: $N - \alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ even.
Since $\alpha - \beta < d_1d_2$ and $l < d_1d_2$ then $\alpha - \beta \le d_1d_2 - l$ and $\alpha - \beta \ge d_1d_2 - l$ are both possible making $N \le \beta + d_1d_2$ or $N \ge \beta + d_1d_2$ which gives us Cases 1A and 1B.

Case 1A: $N - \alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ even, $N - \beta \le d_1d_2$
We have $\gamma = \alpha$, $\delta = \beta + d_1d_2$ and the chain,

$$\beta = \delta - d_1d_2 < \alpha = \gamma < N = \alpha + l \le< \beta + d_1d_2 = \delta < \alpha + d_1d_2 = \gamma + d_1d_2$$

from which, $SR(\alpha) = 1,\ SR(\beta) = 1,\ SR(\gamma) = -1,\ SR(\delta) = 0$ can be entered in the Table below. Then,

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{l}{d_1d_2}\right] = 0, \qquad \left[\frac{N-\gamma}{d_1d_2}\right] = \left[\frac{N-\alpha}{2d_1d_2}\right] = 0$$

From the chain above we have the subchains,

$$\beta < N \le \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0, \qquad \delta - d_1d_2 < N\delta \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] - 1$$

All of these results have been entered in the Table below, the signs on the entries in the γ and δ rows being changed as discussed above.

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	β	1	0	
γ	γ	γ	−1		0
δ	0	0	0		1
Total			1		1

Table 21 - Case 1A: $N-\alpha=l,\ l<d_1d_2,\ \alpha$ even, $\alpha>\beta, N-\beta<d_1d_2$

Case 1B: $N-\alpha=l,\ l<d_1d_2, \alpha>\beta, \alpha$ even, $N-\beta\geq d_1d_2$
We have the chain,

$$\beta<\alpha=\gamma<\beta+d_1d_2=\delta\leq N=\alpha+l<\alpha+d_1d_2=\gamma+d_1d_2<\beta+2d_1d_2=\delta+d_1d_2$$

giving $SR(\alpha)=1,\ SR(\beta)=2,\ SR(\gamma)=-1,\ SR(\delta)=-1$.
Again, $\left[\dfrac{N-\alpha}{d_1d_2}\right]=\left[\dfrac{l}{d_1d_2}\right]=0$ and since $\gamma=\alpha$, $\left[\dfrac{N-\gamma}{2d_1d_2}\right]=0$.
From the chain above we have,

$$\beta+d_1d_2\leq N<\beta+2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right]=1, \qquad \delta\leq N<\delta+d_1d_2 \Rightarrow \left[\frac{N-\delta}{d_1d_2}\right]=0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	$\beta+d_1d_2$	2	1	
γ	γ	γ	−1		0
δ	δ	δ	−1		0
Total			1		1

Table 22 - Case 1B: $N-\alpha=l, l<d_1d_2,\ \alpha$ even, $\alpha>\beta, N-\beta\geq d_1d_2$

Case 2: $N-\alpha=l,\ l<d_1d_2, \alpha<\beta, \alpha$ even.

It is not possible for $N-\beta\geq d_1d_2$ since with $\alpha<\beta$ that would make $N-\alpha>d_1d_2$ contradicting $N-\alpha=l, l<d_1d_2$. So $N-\beta<d_1d_2$.
But we have two possibilities for the position of β on its row, either $\beta<N=\alpha+l$ or $\beta>N=\alpha+l$ giving Cases 2A and 2B below.

Case 2A: $N-\alpha=l,\ l<d_1d_2, \alpha<\beta, \alpha$ even, $\beta<N$
We have the chain,

$$\alpha=\gamma<\beta=\delta-d_1d_2<N=\alpha+l<\alpha+d_1d_2<\beta+d_1d_2=\delta$$

Then $SR(\alpha)=1,\ SR(\beta)=1,\ SR(\gamma)=-1,\ SR(\delta)=0$ and,

$$\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{l}{d_1d_2}\right]=0, \qquad \left[\frac{N-\gamma}{2d_1d_2}\right]=\left[\frac{N-\alpha}{2d_1d_2}\right]=0$$

$$\beta < N < \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0, \quad \delta - d_1d_2 < N < \delta \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = -1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	β	1	0	
γ	γ	γ	-1		0
δ	0	0	0		1
Total			1		1

Table 23 - Case 2A: $\beta < N,\ N-\alpha = l,\ l < d_1d_2,\ \alpha$ even, $\alpha > \beta,\ N-\beta < d_1d_2$

Case 2B: $N-\alpha = l,\ l < d_1d_2, \alpha < \beta, \alpha$ even, $\beta > N$
We have the chain,

$$\beta - d_1d_2 = \delta - 2d_1d_2 < \alpha = \gamma < N = \alpha + l < \beta < \delta = \beta + d_1d_2$$

Then $SR(\alpha) = 1,\ SR(\beta) = 0,\ SR(\gamma) = -1,\ SR(\delta) = 0$ and,

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{l}{d_1d_2}\right] = 0, \qquad \left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{N-\alpha}{2d_1d_2}\right] = 0$$

$$\beta - d_1d_2 < N < \beta \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = -1, \quad \delta - 2d_1d_2 < N < \delta \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = -1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	β	0	-1	
γ	γ	γ	-1		0
δ	0	0	0		1
Total			0		0

Table 24 - Case 2B: $\beta > N, N-\alpha = l, l < d_1d_2,\ \alpha$ even, $\alpha > \beta, N-\beta < d_1d_2$

Case 3: $N-\alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd.

As for Case 1 there are two options for β we label $3A$ and $3B$.

Case 3A: $N-\alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd, $N-\beta \le d_1d_2$
We have $\delta = \beta,\ \gamma = \alpha + d_1d_2$ and,

$$\beta = \delta < \alpha = \gamma - d_1d_2 < N = \alpha + l \le \beta + d_1d_2 = \delta + d_1d_2 < \alpha + d_1d_2 = \gamma$$

Then, $SR(\alpha) = 1,\ SR(\beta) = 1,\ SR(\gamma) = 0,\ SR(\delta) = -1.$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{l}{d_1d_2}\right] = 0, \qquad \gamma - d_1d_2 < N < \gamma \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = -1$$

$$\beta < N \le \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0 \qquad \delta < N < \delta + d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	β	1	0	
γ	0	0	0		1
δ	δ	δ	-1		0
Total			1		1

Table 25 - Case 3A: $N - \alpha = l, l < d_1d_2,\ \alpha$ odd, $\alpha > \beta, N - \beta < d_1d_2$

Case 3B: $N - \alpha = l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd, $N - \beta \ge d_1d_2$
We have the chain,
$\beta = \delta < \alpha = \gamma - d_1d_2 < \beta + d_1d_2 = \delta + d_1d_2$
$\le N = \alpha + l < \alpha + d_1d_2 = \gamma < \beta + 2d_1d_2 = \delta + 2d_1d_2$
Then, $SR(\alpha) = 1,\ SR(\beta) = 2,\ SR(\gamma) = 0,\ SR(\delta) = -1.$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{l}{d_1d_2}\right] = 0, \qquad \gamma - d_1d_2 < N < \gamma \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = -1$$

$$\beta + d_1d_2 < N < \beta + 2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 1 \qquad \delta < N < \delta + d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	$\beta + d_1d_2$	2	1	
γ	0	0	0		1
δ	δ	δ	-1		0
Total			2		2

Table 26 - Case 3B: $N - \alpha = l, l < d_1d_2,\ \alpha$ odd, $\alpha > \beta, N - \beta \ge d_1d_2$

Case 4: $N - \alpha = l,\ l < d_1d_2, \alpha < \beta, \alpha$ odd.

Again it is not possible for $N - \beta \ge d_1d_2$ since that would make $N - \alpha > d_1d_2$, so we must have $N - \beta < d_1d_2$ and then two chains depending on the positioning of β.

Case 4A: $N - \alpha = l,\ l < d_1d_2, \alpha < \beta, \alpha$ odd, $\beta < N$
We have the first chain,

$$\alpha = \gamma - d_1d_2 < \beta = \delta \le N = \alpha + l < \alpha + d_1d_2 = \gamma < \beta + d_1d_2 = \delta + d_1d_2$$

Then, $SR(\alpha) = 1,\ SR(\beta) = 1,\ SR(\gamma) = 0,\ SR(\delta) = -1$, and,

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{l}{d_1d_2}\right] = 0, \qquad \beta \le N < \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0$$
$$\gamma - d_1d_2 < N < \gamma \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = -1, \qquad \delta \le N < \delta - d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	β	1	0	
γ	0	0	0		1
δ	δ	δ	-1		0
Total			1		1

Table 27 - Case 4A: $\beta < N, N - \alpha = l, l < d_1d_2$, α odd, $\alpha < \beta, N - \beta \le d_1d_2$

Case 4B: $N - \alpha = l,\ l < d_1d_2, \alpha < \beta, \alpha$ odd, $\beta \ge N$
We have the second chain,

$$\beta - d_1d_2 = \delta - d_1d_2 < \alpha = \gamma - d_1d_2 < N = \alpha + l \le \beta = \delta < \alpha + d_1d_2 = \gamma$$

Then, $SR(\alpha) = 1,\ SR(\beta) = 0,\ SR(\gamma) = 0,\ SR(\delta) = 0$, and,

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{l}{d_1d_2}\right] = 0, \qquad \beta - d_1d_2 < N \le \beta \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = -1$$
$$\gamma - d_1d_2 < N < \gamma \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = -1, \qquad \delta - d_1d_2 < N \le \delta \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = -1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	α	1	0	
β	β	β	0	-1	
γ	0	0	0		1
δ	δ	δ	0		1
Total			1		1

Table 28 - Case 4B: $\beta > N, N - \alpha = l, l < d_1d_2$, α odd, $\alpha < \beta, N - \beta < d_1d_2$

Case 5: $N - \alpha = d_1d_2$

We must have α odd since N is even and d_1d_2 is odd. So $\gamma = \alpha + d_1d_2$ and $\delta = \beta$. There are two options for $\beta, \beta > \alpha$ or $\beta < \alpha$.

Case 5A: $N - \alpha = d_1d_2$, $\beta < \alpha$ so $N - \beta > d_1d_2$,
We have the SR values in the Table from the chain,

$$\beta = \delta < \alpha = \gamma - d_1d_2 < \beta + d_1d_2 = \delta + d_1d_2 < N = \alpha + d_1d_2 = \gamma < \beta + 2d_1d_2 = \delta + 2d_1d_2$$

and,

$$\left[\frac{N-\alpha}{d_1d_2}\right] = 1, \qquad \beta + d_1d_2 < N < \beta + 2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 1$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{N-N}{2d_1d_2}\right] = 0, \qquad \delta + d_1d_2 < N < \delta + 2d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	$\beta + d_1d_2$	2	1	
γ	γ	γ	-1		0
δ	δ	δ	-1		0
Total			2		2

Table 29 - Case 5A: $N - \alpha = d_1d_2, \beta < \alpha$

Case 5B: $N - \alpha = d_1d_2$, $\beta > \alpha$ so $N - \beta < d_1d_2$.
We have,

$$\alpha < \beta = \delta < N = \alpha + d_1d_2 = \gamma < \beta + d_1d_2 = \delta + d_1d_2$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = 1, \qquad \beta < N < \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{N-N}{2d_1d_2}\right] = 0, \qquad \delta < N < \delta + d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	β	1	0	
γ	γ	γ	-1		0
δ	δ	δ	-1		0
Total			1		1

Table 27 - Case 5B: $N - \alpha = d_1d_2, \beta > \alpha$

Case 6: $N-\alpha = d_1d_2 + l,\ l < d_1d_2,\ \alpha$ even, $\alpha > \beta$

There are two options for β since $\alpha - \beta \le d_1d_2 - l$ and $\alpha - \beta \ge d_1d_2 - l$ can both be true. So both $\beta + 2d_1d_2 \ge N = \alpha + d_1d_2 + l$ and $\beta + 2d_1d_2 \le N = \alpha + d_1d_2 + l$ can be true.

Case 6A: $N-\alpha = d_1d_2 + l,\ l < d_1d_2,\ \alpha$ even, $\alpha > \beta,\ \beta + 2d_1d_2 \le N$
We have,

$$\begin{aligned}\beta < \alpha = \gamma < \beta + d_1d_2 = \delta < \alpha + d_1d_2 = \gamma + d_1d_2 < \beta + 2d_1d_2 = \delta + d_1d_2 \le N = \alpha + d_1d_2 + l\\ < \alpha + 2d_1d_2 = \gamma + 2d_1d_2 < \beta + 3d_1d_2 = \delta + 2d_1d_2\end{aligned}$$

$$\begin{aligned}&\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{d_1d_2+l}{d_1d_2}\right] = 1, \qquad \beta + 2d_1d_2 < N < \beta + 3d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2\\ &\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{d_1d_2+l}{2d_1d_2}\right] = 0, \qquad \delta + d_1d_2 < N < \delta + 2d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	$\beta + 2d_1d_2$	3	2	
γ	γ	γ	−1		0
δ	δ	δ	−1		0
Total			3		3

Table 30 - Case 6A: $N-\alpha = d_1d_2 + l, \alpha\ even,\ \alpha > \beta,\ \beta + 2d_1d_2 \le N$.

Case 6B: $N-\alpha = d_1d_2 + l,\ l < d_1d_2,\ \alpha$ even, $\alpha > \beta,\ \beta + 2d_1d_2 \ge N$.
We have,

$$\begin{aligned}\beta < \alpha = \gamma < \beta + d_1d_2 = \delta < \alpha + d_1d_2 = \gamma + d_1d_2 < N\\ \le \beta + 2d_1d_2 = \delta + d_1d_2 < \alpha + 2d_1d_2 = \gamma + 2d_1d_2 < \delta + 2d_1d_2\end{aligned}$$

$$\begin{aligned}&\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{d_1d_2+l}{d_1d_2}\right] = 1, \qquad \beta + d_1d_2 < N < \beta + 2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 1\\ &\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{d_1d_2+l}{2d_1d_2}\right] = 0, \qquad \delta < N < \delta + d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+d_1d_2$	2	1	
β	β	$\beta+d_1d_2$	2	1	
γ	γ	γ	-1		0
δ	δ	δ	-1		0
Total			2		2

Table 31 - Case 6B: $N-\alpha = d_1d_2+l, \alpha\ even,\ \alpha>\beta,\ \beta+2d_1d_2 \geq N$

Case 7: $N-\alpha = d_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha<\beta$

There are two options for β since $\beta-\alpha \leq d_1d_2-l$ and $\beta-\alpha \geq d_1d_2-l$ can both be true. Acccordingly $N \leq \beta+d_1d_2$ or $N \geq \beta+d_1d_2$.

Case 7A: $N-\alpha = d_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha<\beta,\ N-\beta \leq d_1d_2$
We have,

$$\alpha = \gamma < \beta = \delta - d_1d_2 < \alpha + d_1d_2 = \gamma + d_1d_2 < N = \alpha + d_1d_2 + l \leq \beta + d_1d_2 = \delta$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{d_1d_2+l}{d_1d_2}\right] = 1, \qquad \beta < N < \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{d_1d_2+l}{2d_1d_2}\right] = 0, \qquad \delta - d_1d_2 < N < \delta \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = -1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+d_1d_2$	2	1	
β	β	β	1	0	
γ	γ	γ	-1		0
δ	0	0	0		1
Total			2		2

Table 32 - Case 7A: $N-\alpha = d_1d_2+l,\ \alpha\ even,\ \alpha<\beta,\ N-\beta \leq d_1d_2$

Case 7B: $N-\alpha = d_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha<\beta,\ N-\beta \geq d_1d_2$
We have,

$$\begin{aligned}\alpha = \gamma < \beta < \alpha + d_1d_2 = \gamma + d_1d_2 < \beta + d_1d_2 = \delta \leq N = \alpha + d_1d_2 + l \\ < \alpha + 2d_1d_2 = \gamma + 2d_1d_2 < \beta + 2d_1d_2 = \delta + d_1d_2\end{aligned}$$

$$\alpha + d_1d_2 \le N < \alpha + 2d_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 1, \qquad \beta + d_1d_2 < N < \beta + 2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 1$$
$$\gamma + d_1d_2 \le N < \gamma + 2d_1d_2 \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = 0, \qquad \delta < N < \delta + d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	$\beta + d_1d_2$	2	1	
γ	γ	γ	−1		0
δ	δ	δ	−1		0
Total			2		2

Table 33 - Case 7B: $N - \alpha = d_1d_2 + l,\ \alpha\ even,\ \alpha < \beta,\ N - \beta \ge d_1d_2$

Case 8: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd.

As for Case 6, there are two options for β.
Case 8A: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd, $\beta + 2d_1d_2 \le N$.
We have,

$$\beta = \delta < \alpha < \beta + d_1d_2 < \alpha + d_1d_2 = \gamma < \beta + 2d_1d_2 = \delta + 2d_1d_2 \le N = \alpha + d_1d_2 + l$$
$$< \alpha + 2d_1d_2 = \gamma + d_1d_2 < \beta + 3d_1d_2 = \delta + 3d_1d_2$$

$$\alpha + d_1d_2 \le N < \alpha + 2d_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 1, \qquad \beta + 2d_1d_2 \le N < \beta + 3d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2$$
$$\gamma < N < \gamma + d_1d_2 \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = 0, \qquad \delta + 2d_1d_2 < N < \delta + 3d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	$\beta + 2d_1d_2$	3	2	
γ	γ	γ	−1		0
δ	δ	$\delta + 2d_1d_2$	−2		−1
Total			2		2

Table 34 - Case 8A: $N - \alpha = d_1d_2 + l,\ \beta + 2d_1d_2 \le N,\ \alpha > \beta,\ \alpha$ odd

Case 8B: $N - \alpha = d_1d_2 + l,\ l < d_1d_2, \alpha > \beta, \alpha$ odd,$\beta + 2d_1d_2 \ge N$.
We have,

$$\beta = \delta < \alpha < \beta + d_1d_2 < \alpha + d_1d_2 = \gamma$$
$$\le N = \alpha + d_1d_2 + l < \alpha + 2d_1d_2 = \gamma + d_1d_2 < \beta + 2d_1d_2 = \delta + 2d_1d_2$$

$$\alpha + d_1d_2 \le N < \alpha + 2d_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 1,\ \beta + d_1d_2 < N < \beta + 2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 1$$
$$\gamma < N < \gamma + d_1d_2 \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = 0,\ \delta + d_1d_2 < N < \delta + 2d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	$\beta + d_1d_2$	2	1	
γ	γ	γ	−1		0
δ	δ	δ	−1		0
Total			2		2

Table 35 - Case 8B: $N - \alpha = d_1d_2 + l,\ \alpha > \beta,\ \alpha\ even,\ \beta + 2d_1d_2 \ge N.$

Case 9: $N - \alpha = d_1d_2 + l,\ l < d_1d_2,\ \alpha < \beta,\ \alpha$ odd.

As in Cases 6, 7 and 8, there are two options for β.

Case 9A: $N - \alpha = d_1d_2 + l,\ l < d_1d_2,\ \alpha < \beta,\ \alpha$ odd, $N - \beta \le d_1d_2$
We have,

$$\alpha < \beta = \delta < \alpha + d_1d_2 = \gamma < N = \alpha + d_1d_2 + l = \gamma + l < \beta + d_1d_2 = \delta + d_1d_2.$$

$$\alpha + d_1d_2 \le N < \alpha + 2d_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 1,\ \beta < N < \beta + d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 0$$
$$\gamma \le N < \gamma + l \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = 0,\ \delta < N < \delta + d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + d_1d_2$	2	1	
β	β	β	1	0	
γ	γ	γ	−1		0
δ	δ	δ	−1		0
Total			1		1

Table 36 - Case 9A: $N - \alpha = d_1d_2 + l,\ N - \beta \le d_1d_2,\ \alpha < \beta, \alpha$ odd

Case 9B: $N - \alpha = d_1d_2 + l,\ l < d_1d_2,\ \alpha < \beta,\ \alpha$ odd, $N - \beta \ge d_1d_2$
We have,

$$\alpha < \beta = \delta < \alpha + d_1d_2 = \gamma < \beta + d_1d_2 = \delta + d_1d_2 < N = \alpha + d_1d_2 + l$$
$$< \alpha + 2d_1d_2 = \gamma + d_1d_2 < \beta + 2d_1d_2 = \delta + 2d_1d_2$$

$$\alpha + d_1d_2 \le N < \alpha + 2d_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 1,\ \beta + d_1d_2 < N < \beta + 2d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 1$$
$$\gamma \le N < \gamma + d_1d_2 \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = 0,\ \delta + d_1d_2 < N < \delta + 2d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = 0$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+d_1d_2$	2	1	
β	β	$\beta+d_1d_2$	2	1	
γ	γ	γ	-1		0
δ	δ	δ	-1		0
Total			2		2

Table 37 - Case 9B: $N-\alpha = d_1d_2 + l,\ N-\beta \ge d_1d_2,\ \alpha < \beta, \alpha$ odd

Case 10: $N-\alpha = 2kd_1d_2$, α even, $\alpha > \beta$, $k \ge 1$
We have,

$$\begin{aligned}
&\beta < \alpha = \gamma < \beta + d_1d_2 = \delta < \alpha + d_1d_2 \dots \\
&< \beta + 2kd_1d_2 = \delta + (2k-1)d_1d_2 < \alpha + 2kd_1d_2 = \gamma + 2kd_1d_2 = N \\
&< \beta + (2k+1)d_1d_2 = \delta + 2kd_1d_2 < \alpha + (2k+1)d_1d_2 = \gamma + (2k+1)d_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2}{d_1d_2}\right] = 2k,\ \beta + 2kd_1d_2 < N < \beta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = k,\ \delta + (2k-1)d_1d_2 < N < \delta + 2kd_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k-1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+(k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k+1$		$2k+1$

Table 38 - Case 10: $N-\alpha = 2kd_1d_2$, α even, $\alpha > \beta$ $k \ge 1$

Case 11: $N-\alpha = 2kd_1d_2$, α even, $\alpha < \beta$, $k > 1$.
We have,

$$\begin{aligned}
&\alpha = \gamma < \beta < \alpha + d_1d_2 = \gamma + d_1d_2 < \beta + d_1d_2 = \delta \dots \\
&< \alpha + (2k-1)d_1d_2 < \beta + (2k-1)d_1d_2 = \delta + (k-1)2d_1d_2 \dots \\
&< \alpha + 2kd_1d_2 = \gamma + 2kd_1d_2 = N < \beta + 2kd_1d_2 = \delta + (2k-1)d_1d_2
\end{aligned}$$

$$N - \alpha = 2kd_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 2k,$$
$$\beta + (2k-1)d_1d_2 < N < \beta + 2kd_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k-1$$
$$N - \gamma = 2kd_1d_2 \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = k,$$
$$\delta + (k-1)2d_1d_2 < N < \delta + (2k-1)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k-1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+(2k-1)d_1d_2$	$2k$	$2k-1$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+(k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k$		$2k$

Table 39 - Case 11: $N-\alpha = 2kd_1d_2$, α even, $\alpha < \beta$, $k \geq 1$.

Case 12: $N - \alpha = 2kd_1d_2 + l,\ l < d_1d_2$, α even, $\alpha > \beta, k \geq 1$
We can have $\beta + 2kd_1d_2 + d_1d_2 \leq \alpha + 2kd_1d_2 + l = N$ since then $\alpha - \beta \geq d_1d_2 - l$ which can be true, but we can also have $N = \alpha + 2kd_1d_2 + l \leq \beta + (2k+1)d_1d_2$ from which $\alpha - \beta \leq d_1d_2 - l$, which can also be true. So there are two options for the position of the final entry on the β line.

Case 12A: $N - \alpha = 2kd_1d_2 + l,\ l < d_1d_2$, α even, $\alpha > \beta$, $\beta + (2k+1)d_1d_2 \leq N$
We have,

$$\beta < \alpha = \gamma < \beta + d_1d_2 = \delta \ldots$$
$$\beta + 2kd_1d_2 < \alpha + 2kd_1d_2 = \gamma + 2kd_1d_2 < \beta + (2k+1)d_1d_2 = \delta + 2kd_1d_2 \leq N$$
$$= \alpha + 2kd_1d_2 + l < \alpha + 2kd_1d_2 + d_1d_2 = \gamma + (2k+1)d_1d_2$$
$$< \beta + (2k+2)d_1d_2 = \delta + (2k+1)d_1d_2$$

$$\alpha + 2kd_1d_2 < N < \alpha + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right] = 2k,$$
$$\beta + (2k+1)d_1d_2 \leq N < \beta + (2k+2)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k+1$$
$$\gamma + 2kd_1d_2 < N < \gamma + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right] = k,$$
$$\delta + 2kd_1d_2 \leq N < \delta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+(2k+1)d_1d_2$	$2k+2$	$2k+1$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+1$		$2k+1$

Table 40 - Case 12A: $N-\alpha=2kd_1d_2+l,\ \alpha\ even,\alpha>\beta,\ \beta+(2k+1)d_1d_2\le N$

Case 12B: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\alpha\ even,\alpha>\beta,\ \beta+(2k+1)d_1d_2\ge N.$
We have,

$$\begin{aligned}
&\beta<\alpha=\gamma<\beta+d_1d_2=\delta\dots\\
&<\beta+(2k-1)d_1d_2=\delta+(k-1)2d_1d_2<\alpha+(2k-1)d_1d_2=\gamma+(2k-1)d_1d_2\\
&<\beta+2kd_1d_2=\delta+(2k-1)d_1d_2<\alpha+2kd_1d_2=\gamma+2kd_1d_2\\
&<N=\alpha+2kd_1d_2+l\le\beta+(2k+1)d_1d_2=\delta+2kd_1d_2\\
&<\alpha+(2k+1)d_1d_2=\gamma+(2k+1)d_1d_2
\end{aligned}$$

$$\begin{aligned}
\alpha+2kd_1d_2<N<\alpha+(2k+1)d_1d_2 &\Rightarrow \left[\frac{N-\alpha}{d_1d_2}\right]=2k,\\
\beta+2kd_1d_2<N\le\beta+(2k+1)d_1d_2 &\Rightarrow \left[\frac{N-\beta}{d_1d_2}\right]=2k\\
\gamma+2kd_1d_2<N<\gamma+(2k+1)d_1d_2 &\Rightarrow \left[\frac{N-\gamma}{2d_1d_2}\right]=k,\\
\delta+(2k-1)d_1d_2<N\le\delta+2kd_1d_2 &\Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right]=k-1
\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+(k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k+1$		$2k+1$

Table 41 - Case 12B: $N-\alpha=2kd_1d_2+l,\ \alpha\ even,\ \alpha>\beta,\ \beta+(2k+2)d_1d_2\ge N$

Case 13: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha<\beta$,
Since $\beta-\alpha<l$ and $\beta-\alpha\ge l$ are both possible, there are two options for the final entry on the β row.

Case 13A: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha>\beta,\ \beta+2kd_1d_2\le N$
We have,

$$\begin{aligned}&\alpha=\gamma<\beta<\alpha+d_1d_2<\beta+d_1d_2=\delta\ldots\\&\beta+(2k-1)d_1d_2<\delta+(k-1)2d_1d_2<\alpha+2kd_1d_2=\gamma+2kd_1d_2\\&<\beta+2kd_1d_2=\delta+(2k-1)d_1d_2\le N=\alpha+2kd_1d_2+l<\alpha+(2k+1)d_1d_2\\&=\gamma+(2k+1)d_1d_2<\beta+(2k+1)d_1d_2=\delta+2kd_1d_2\end{aligned}$$

$$\begin{aligned}&\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{d_1d_2}\right]=2k,\qquad\left[\frac{N-\gamma}{2d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{2kd_1d_2}\right]=k\\&\beta+2kd_1d_2\le N<\beta+(2k+1)d_1d_2\Rightarrow\left[\frac{N-\beta}{d_1d_2}\right]=2k\\&\delta+(2k-1)d_1d_2\le N<\delta+2kd_1d_2\Rightarrow\left[\frac{N-\delta}{2d_1d_2}\right]=k-1\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+(k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k+1$		$2k+1$

Table 42 - Case 13A: $N-\alpha=2kd_1d_2+l,\ \alpha$ *even*, $\alpha<\beta,\ \beta+2kd_1d_2\le N$

Case 13B: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha>\beta,\ \beta+2kd_1d_2\ge N$
We have,

$$\begin{aligned}&\alpha=\gamma<\beta<\alpha+d_1d_2<\beta+d_1d_2=\delta\ldots\\&<\alpha+(2k-1)d_1d_2<\beta+(2k-1)d_1d_2=\delta+(k-1)2d_1d_2\\&<\alpha+2kd_1d_2=\gamma+2kd_1d_2<N=\alpha+2kd_1d_2+l\le\beta+2kd_1d_2\\&=\delta+(2k-1)d_1d_2<\alpha+(2k+1)d_1d_2=\gamma+(2k+1)d_1d_2\end{aligned}$$

$$\begin{aligned}&\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{d_1d_2}\right]=2k,\qquad\left[\frac{N-\gamma}{2d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{2kd_1d_2}\right]=k\\&\beta+(2k-1)d_1d_2<N<\beta+2kd_1d_2\Rightarrow\left[\frac{N-\beta}{d_1d_2}\right]=2k-1\\&\delta+(k-1)2d_1d_2<N<\delta+(2k-1)d_1d_2\Rightarrow\left[\frac{N-\delta}{2d_1d_2}\right]=k-1\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+(2k-1)d_1d_2$	$2k$	$2k-1$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+(k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k$		$2k$

Table 43 - Case 13B: $N-\alpha=2kd_1d_2+l,\ \alpha\ even,\ \alpha<\beta,\ \beta+2kd_1d_2\geq N$

Case 14: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2$, α odd, $\alpha>\beta$
As for Case 13 we have two options on the β row.

Case 14A: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2$, α odd, $\alpha>\beta$, $\beta+(2k+1)d_1d_2\leq N$
We have,

$$\begin{aligned}&\beta=\delta<\alpha<\beta+d_1d_2<\alpha+d_1d_2=\gamma\dots\\&<\alpha+(2k-1)d_1d_2=\gamma+(k-1)2d_1d_2<\beta+2kd_1d_2=\delta+2kd_1d_2\\&<\alpha+2kd_1d_2=\gamma+(2k-1)d_1d_2<\beta+(2k+1)d_1d_2\\&=\delta+(2k+1)d_1d_2\leq N=\alpha+2kd_1d_2+l<\alpha+(2k+1)d_1d_2\\&=\gamma+2kd_1d_2<\beta+(2k+2)d_1d_2=\delta+(k+1)2d_1d_2\end{aligned}$$

$$\begin{aligned}&\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{d_1d_2}\right]=2k\\&\left[\frac{N-\gamma}{2d_1d_2}\right]=\left[\frac{N-\alpha-d_1d_2}{2d_1d_2}\right]=\left[\frac{2kd_1d_2+l-d_1d_2}{2d_1d_2}\right]=k-1\\&\beta+(2k+1)d_1d_2\leq N<\beta+(2k+2)d_1d_2\Rightarrow\left[\frac{N-\beta}{d_1d_2}\right]=2k+1\\&\delta+(2k+1)d_1d_2\leq N<\delta+(2k+2)d_1d_2\Rightarrow\left[\frac{N-\delta}{2d_1d_2}\right]=k\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+(2k+1)d_1d_2$	$2k+2$	$2k+1$	
γ	γ	$\gamma+(k-1)2d_1d_2$	$-k$		$-(k-1)$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+2$		$2k+2$

Table 44 - Case 14A: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2$, α odd, $\alpha>\beta$, $\beta+(2k+1)d_1d_2\leq N$

Case 14B: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2$, α odd, $\alpha>\beta$, $\beta+(2k+1)d_1d_2\geq N$

$$\beta = \delta < \alpha < \beta + d_1d_2 < \alpha + d_1d_2 \dots$$
$$< \alpha + (2k-1)d_1d_2 = \gamma + (k-1)2d_1d_2 < \beta + 2kd_1d_2 = \delta + 2kd_1d_2$$
$$< \alpha + 2kd_1d_2 < N = \alpha + 2kd_1d_2 + l \le \beta + (2k+1)d_1d_2 = \delta + (2k+1)d_1d_2$$
$$< \alpha + (2k+1)d_1d_2 = \gamma + (2k+1)d_1d_2$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + l}{d_1d_2}\right] = 2k$$
$$\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{N-\alpha-d_1d_2}{2d_1d_2}\right] = \left[\frac{2kd_1d_2 + l - d_1d_2}{2d_1d_2}\right] = k-1$$
$$\beta + 2kd_1d_2 < N < \beta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k$$
$$\delta + 2kd_1d_2 < N < \delta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + 2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta + 2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma + (k-1)2d_1d_2$	$-k$		$-(k-1)$
δ	δ	$\delta + 2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+1$		$2k+1$

Table 45 - Case 14B:$N - \alpha = 2kd_1d_2 + l,\ l < d_1d_2$, α odd, $\alpha > \beta$, $\beta + (2k+1)d_1d_2 \ge N$

Case 15: $N - \alpha = 2kd_1d_2 + l,\ l < d_1d_2$, α odd, $\beta > \alpha$
We have two possibilities for the last entry on the β row.

Case 15A: $N - \alpha = 2kd_1d_2 + l,\ l < d_1d_2$, α odd, $\beta > \alpha$, $\beta + 2kd_1d_2 \le N$.
We have,

$$\alpha < \beta = \delta < \alpha + d_1d_2 = \gamma < \dots$$
$$< \alpha + (2k-1)d_1d_2 = \gamma + (k-1)2d_1d_2 < \beta + (2k-1)d_1d_2 < \alpha + 2kd_1d_2$$
$$= \gamma + (2k-1)d_1d_2 < \beta + 2kd_1d_2 = \delta + 2kd_1d_2 \le N = \alpha + 2kd_1d_2 + l$$
$$< \alpha + (2k+1)d_1d_2 = \gamma + 2kd_1d_2 < \beta + (2k+1)d_1d_2 = \delta + (2k+1)d_1d_2$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + l}{d_1d_2}\right] = 2k \qquad \left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2kd_1d_2 + l - d_1d_2}{2d_1d_2}\right] = k-1$$
$$\beta + 2kd_1d_2 < N < \beta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right]2k$$
$$\delta + 2kd_1d_2 < N < \delta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma+(k-1)2d_1d_2$	$-k$		$-(k-1)$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+1$		$2k+1$

Table 46 - Case 15A: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\ \alpha$ odd, $\beta>\alpha,\ \beta+2kd_1d_2\le N$

Case 15B: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\ \alpha$ odd, $\beta>\alpha,\ N\le\beta+2kd_1d_2$
We have,

$$\begin{aligned}&\alpha<\beta=\delta<\alpha+d_1d_2=\gamma\dots\\&<\beta+(2k-2)d_1d_2=\delta+(k-1)2d_1d_2<\alpha+(2k-1)d_1d_2\\&=\gamma+(k-1)2d_1d_2<\beta+(2k-1)d_1d_2=\delta+(2k-1)d_1d_2\\&<\alpha+2kd_1d_2<N\le\beta+2kd_1d_2=\delta+2kd_1d_2\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{d_1d_2}\right]=2k,\qquad\left[\frac{N-\gamma}{2d_1d_2}\right]=\left[\frac{2kd_1d_2+l-d_1d_2}{2d_1d_2}\right]=k-1$$

$$\beta+(2k-1)d_1d_2<N\le\beta+2kd_1d_2\Rightarrow\left[\frac{N-\beta}{d_1d_2}\right]=2k-1$$

$$\delta+(2k-1)d_1d_2<N\le\delta+2kd_1d_2\Rightarrow\left[\frac{N-\delta}{2d_1d_2}\right]k-1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2$	$2k+1$	$2k$	
β	β	$\beta+(2k-1)d_1d_2$	$2k$	$2k-1$	
γ	γ	$\gamma+(k-1)2d_1d_2$	$-k$		$-(k-1)$
δ	δ	$\delta+(k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k+1$		$2k+1$

Table 47 - Case 15B: $N-\alpha=2kd_1d_2+l,\ l<d_1d_2,\ \alpha$ odd, $\beta>\alpha,\ \beta+2kd_1d_2\ge N$

Case 16: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha>\beta$

Now $\beta+2kd_1d_2+2d_1d_2<N=\alpha+2kd_1d_2+d_1d_2+l\Rightarrow\alpha-\beta>d_1d_2-l$ which can be true.
But we can also have $N=\alpha+2kd_1d_2+d_1d_2+l<\beta+2kd_1d_2+2d_1d_2\Rightarrow\alpha-\beta<d_1d_2-l$ which can also be true.
There are therefore two possibilities for the last β line entry.

Case 16A: $N-\alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2$, α even, $\alpha > \beta$, $\beta + 2kd_1d_2 + 2d_1d_2 \le N$. We have,

$$\begin{aligned}
&\beta < \alpha = \gamma < \beta + d_1d_2 = \delta < \dots\\
&\alpha + 2kd_1d_2 = \gamma + 2kd_1d_2 < \beta + 2d_1d_2 + d_1d_2 = \delta + 2kd_1d_2\\
&< \alpha + 2kd_1d_2 + d_1d_2 < \beta + 2kd_1d_2 + 2d_1d_2 = \delta + 2kd_1d_2 + d_1d_2\\
&\le N = \alpha + 2kd_1d_2 + d_1d_2 + l < \beta + (2k+3)d_1d_2 = \delta + (2k+2)d_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{d_1d_2}\right] = 2k+1, \qquad \left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{2d_1d_2}\right] = k.$$

$$\beta + (2k+2)d_1d_2 \le N < \beta + (2k+3)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k+2$$

$$\delta + (2k+1)d_1d_2 < N < \delta + (2k+2)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + 2kd_1d_2 + d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta + 2kd_1d_2 + 2d_1d_2$	$2k+3$	$2k+2$	
γ	γ	$\gamma + 2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta + 2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+3$		$2k+3$

Table 48 - Case 16A: $N-\alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2$, α even, $\alpha > \beta$, $\beta + 2kd_1d_2 + d_1d_2 \le N$

Case 16B: $N-\alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2$, α even, $\alpha > \beta$, $N \le \beta + 2kd_1d_2 + 2d_1d_2$ We have,

$$\begin{aligned}
&\beta < \alpha < \beta + d_1d_2 = \delta \dots\\
&< \alpha + 2kd_1d_2 = \gamma + 2kd_1d_2 < \beta + 2kd_1d_2 + d_1d_2 = \delta + 2kd_1d_2 < \alpha + 2kd_1d_2 + d_1d_2\\
&= \gamma + 2kd_1d_2 + d_1d_2 < N = \alpha + 2kd_1d_2 + l \le \beta + 2kd_1d_2 + 2d_1d_2 = \delta + (2k+1)d_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{d_1d_2}\right] = 2k+1, \qquad \left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{N-\alpha}{2d_1d_2}\right] = k.$$

$$\beta + (2k+1)d_1d_2 < N \le \beta + (2k+2)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k.$$

$$\delta + 2kd_1d_2 < N \le \delta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k-1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta+2kd_1d_2+d_1d_2$	$2k+2$	$2k$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-(k-1)$
Total			$2k+2$		$2k+2$

Table 49 - Case 16B: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\alpha>\beta,\ N\leq\beta+2kd_1d_2+2d_1d_2$

Case 17: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2,\ \alpha$ even, $\beta>\alpha$
There are two possibilites for the last entry on the β row.

Case 17A: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2, \alpha\ even, \beta>\alpha,$
$\beta+2kd_1d_2+d_1d_2\leq N.$
We have,

$$\begin{aligned}
&\alpha=\gamma<\beta<\alpha+d_1d_2=\gamma+d_1d_2<\beta+d_1d_2=\delta\ldots\\
&<\alpha+2kd_1d_2=\gamma+2kd_1d_2<\beta+2kd_1d_2=\delta+(2k-1)<\alpha+2kd_1d_2+d_1d_2\\
&<\beta+2kd_1d_2+d_1d_2=\delta+2kd_1d_2\leq N=\alpha+2kd_1d_2+d_1d_2+l\\
&<\beta+(2k+2)d_1d_2=\delta+(2k+1)d_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{2kd_1d_2+d_1d_2+l}{d_1d_2}\right]=2k+1,\qquad\left[\frac{N-\gamma}{2d_1d_2}\right]=\left[\frac{N-\alpha}{2d_1d_2}\right]=k.$$

$$\beta+(2k+1)d_1d_2\leq N<\beta+(2k+2)d_1d_2\Rightarrow\left[\frac{N-\beta}{d_1d_2}\right]=2k+1$$

$$\delta+2kd_1d_2\leq N<\delta+(2k+1)d_1d_2\Rightarrow\left[\frac{N-\delta}{2d_1d_2}\right]=k$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+2$		$2k+2$

Table 50 - Case 17A: $N-\alpha=2kd_1d_2+d_1d_2+l,\ \alpha\ even,\ \beta>\alpha,\ \beta+2kd_1d_2+d_1d_2\leq N.$

Case 17B: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2, \alpha\ even, \beta>\alpha, \beta+(2k+1)d_1d_2\geq N.$

We have,

$$\begin{aligned}
&\alpha = \gamma < \beta < \alpha + d_1d_2 < \beta + d_1d_2 = \delta \ldots \\
&< \beta + (2k-1)d_1d_2 = \delta + (k-1)2d_1d_2 < \alpha + 2kd_1d_2 = \gamma + 2kd_1d_2 \\
&< \beta + 2kd_1d_2 = \delta + (2k-1)d_1d_2 < \alpha + 2kd_1d_2 + d_1d_2 \\
&= \gamma + (2k+1)d_1d_2 \le N = \alpha + 2kd_1d_2 + d_1d_2 + l \\
&\le \beta + 2kd_1d_2 + d_1d_2 = \delta + 2kd_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{d_1d_2}\right] = 2k+1, \quad \left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{2d_1d_2}\right] = k$$

$$\beta + 2kd_1d_2 < N \le \beta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k$$

$$\delta + (2k-1)d_1d_2 < N \le \delta + 2kd_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k-1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + 2kd_1d_2 + d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta + 2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma + 2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta + (k-1)2d_1d_2$	$-k$		$-(k-1)$
Total			$2k+2$		$2k+2$

Table 51 - Case 17B: $\beta + 2kd_1d_2 + d_1d_2 \ge N$.

Case 18: $N - \alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2, \alpha\ odd, \alpha > \beta$
There are two possibilities for the last entry on the β row.

Case 18A: $N - \alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2,\ \alpha$ odd, $\alpha > \beta,\ \beta + 2kd_1d_2 + 2d_1d_2 \le N$.
We have,

$$\begin{aligned}
&\beta = \delta < \alpha < \beta + d_1d_2 < \alpha + d_1d_2 = \gamma \ldots \\
&\quad < \alpha + 2kd_1d_2 + d_1d_2 = \gamma + 2kd_1d_2 < \beta + 2kd_1d_2 + 2d_1d_1 \\
&= \delta + 2kd_1d_2 + 2d_1d_2 \le N < \beta + (2k+3)d_1d_2 = \delta + (2k+3)d_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{d_1d_2}\right] = 2k+1, \quad \left[\frac{N-\gamma}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + l}{2d_1d_2}\right] = k$$

$$\beta + (2k+2)d_1d_2 \le N < \beta + (2k+3)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k+2$$

$$\delta + (k+1)2d_1d_2 \le N < \delta + (2k+3)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k+1$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta+2kd_1d_2+2d_1d_2$	$2k+3$	$2k+2$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+2kd_1d_2+2d_1d_2$	$-(k+2)$		$-(k+1)$
Total			$2k+2$		$2k+2$

Table 52 - Case 18A: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2,\ \alpha\ odd, \alpha>\beta,\ \beta+2kd_1d_2+d_1d_2\leq N.$

Case 18B: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2$, α odd, $\alpha>\beta$, $\beta+2kd_1d_2+2d_1d_2\geq N$.
We have,

$$\begin{aligned}
&\beta=\delta<\alpha<\beta+d_1d_2<\alpha+d_1d_2=\gamma\dots\\
&\ <\beta+2kd_1d_2=\delta+2kd_1d_2<\alpha+2kd_1d_2<\beta+2kd_1d_2+d_1d_2\\
&=\delta+2kd_1d_2+d_1d_2<\alpha+2kd_1d_2+d_1d_2=\gamma+2kd_1d_2\\
&<N\leq\beta+(2k+2)d_1d_2=\delta+(2k+2)d_1d_2
\end{aligned}$$

$$\left[\frac{N-\alpha}{d_1d_2}\right]=\left[\frac{2kd_1d_2+d_1d_2+l}{d_1d_2}\right]=2k+1,\qquad \left[\frac{N-\gamma}{d_1d_2}\right]=\left[\frac{2kd_1d_2+l}{2d_1d_2}\right]=k$$

$$\beta+(2k+1)d_1d_2<N\leq\beta+(2k+2)d_1d_2\Rightarrow\left[\frac{N-\beta}{d_1d_2}\right]=2k+1$$

$$\delta+(2k+1)d_1d_2<N\leq\delta+(2k+2)d_1d_2\Rightarrow\left[\frac{N-\delta}{2d_1d_2}\right]=k$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+2$		$2k+2$

Table 53 - Case 18B: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2$, α odd, $\alpha>\beta$, $\beta+2kd_1d_2+d_1d_2\geq N$.

Case 19: $N-\alpha=2kd_1d_2+d_1d_2+l,\ \alpha\ odd,\ \beta>\alpha$
There are two possibilities for the β line.

Case 19A: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2$, α odd, $\beta>\alpha$, $\beta+2kd_1d_2+d_1d_2\leq N$

We have,

$$\begin{aligned}
&\alpha < \beta = \delta < \alpha + d_1d_2 = \gamma \ldots \\
&< \beta + 2kd_1d_2 = \delta + 2kd_1d_2 < \alpha + 2kd_1d_2 + d_1d_2 = \gamma + 2kd_1d_2 \\
&< \beta + 2kd_1d_2 + d_1d_2 \le N = \alpha + 2kd_1d_2 + d_1d_2 + l \\
&< \beta + (2k+2)d_1d_2 = \delta + (2k+2)d_1d_2
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{d_1d_2}\right] = 2k+1 \\
&\left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2kd_1d_2 + l}{2d_1d_2}\right] = k \\
&\beta + (2k+1)d_1d_2 < N < \beta + (2k+2)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k+1 \\
&\delta + (2k+1)d_1d_2 < N < \delta + (2k+2)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k
\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\frac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha + 2kd_1d_2 + d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta + 2kd_1d_2 + d_1d_2$	$2k+2$	$2k+1$	
γ	γ	$\gamma + 2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta + 2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+2$		$2k+2$

Table 54 - Case 19A: $N-\alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2,\ \alpha$ odd, $\alpha > \beta,\ \beta + 2kd_1d_2 + d_1d_2 \le N$

Case 19B: $N-\alpha = 2kd_1d_2 + d_1d_2 + l,\ l < d_1d_2,\ \alpha$ odd, $\beta > \alpha,\ N \le \beta + 2kd_1d_2 + d_1d_2$
We have,

$$\begin{aligned}
&\alpha < \beta = \delta < \alpha + d_1d_2 = \gamma \ldots \\
&< \beta + 2kd_1d_2 = \delta + 2kd_1d_2 < \alpha + 2kd_1d_2 + d_1d_1 = \gamma + 2kd_1d_2 \\
&\le N \le \beta + (2k+1)d_1d_2 = \delta + (2k+1)d_1d_2
\end{aligned}$$

$$\begin{aligned}
&\left[\frac{N-\alpha}{d_1d_2}\right] = \left[\frac{2kd_1d_2 + d_1d_2 + l}{d_1d_2}\right] = 2k+1, \quad \left[\frac{N-\gamma}{2d_1d_2}\right] = \left[\frac{2kd_1d_2 + l}{2d_1d_2}\right] = k \\
&\beta + 2kd_1d_2 < N \le \beta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\beta}{d_1d_2}\right] = 2k \\
&\delta + 2kd_1d_2 < N \le \delta + (2k+1)d_1d_2 \Rightarrow \left[\frac{N-\delta}{2d_1d_2}\right] = k
\end{aligned}$$

x_1	First entry	Last entry	SR	$\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$	$\mu(2d_1d_2)\left[\dfrac{N-x_1}{2d_1d_2}\right]$
α	α	$\alpha+2kd_1d_2+d_1d_2$	$2k+2$	$2k+1$	
β	β	$\beta+2kd_1d_2$	$2k+1$	$2k$	
γ	γ	$\gamma+2kd_1d_2$	$-(k+1)$		$-k$
δ	δ	$\delta+2kd_1d_2$	$-(k+1)$		$-k$
Total			$2k+1$		$2k+1$

Table 55 - Case 19B: $N-\alpha=2kd_1d_2+d_1d_2+l,\ l<d_1d_2,\ \alpha\ odd,\ \alpha<\beta,\ \beta+2kd_1d_2+d_1d_2\geq N.$

In all quartets where $\alpha > N$, the four SR sums add to the same value as the two $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$ values and the two $\mu(2d_1d_2)\left[\dfrac{N-x_1}{2d_1d_2}\right]$ values. □

Our Counting Theorem is proved. □

3.3.6 Conclusion

Lemmas 7, 8, 9, 10 and 11 have proved that the four SR sums of all possible quartets in an expanded Table 4 have the same total value as that of the four respective $\mu(d_1d_2)\left[\dfrac{N-x_1}{d_1d_2}\right]$ values so, for each value of d_1d_2, we can replace one with the other.

Since the sum of all the SR values is a count of the number of the upper twin primes less than N where N is in the interval $(p_\theta, p_{\theta+1}^2)$ we have proved,

Theorem 12. *In the primod-p_θ number system, the number of primods less than an even integer N with no p-digits equal to 0 or $2(\text{mod } p)$ is given by:*

$$T(p_\theta, N, 2) = \sum^{*} \mu(d_1d_2)\left[\frac{N-x_1}{d_1d_2}\right]$$

where $\overset{*}{\sum}$ is a sum with,

(a) p_θ *is the largest prime less than* $\sqrt{N}$

(b) d_1 *is the product of one or more elements of the set* $\{2,3,\ldots,p_\theta\}\cup\{1\}$ *, where* $2,3,\ldots p_\theta$ *are consecutive primes*

(c) d_2 *is the product of one or more elements of the set* $\{3,\ldots,p_\theta\}\cup\{1\}$ *so that* $(d_2,2)=1$.

(d) $(d_1,d_2)=1$

(e) the sum is over all possible values of d_1 *and* d_2,

(f) μ *is the Möbius function and* $[x]$ *the greatest integer function*

(g) x_1 *is the least non negative solution of the system of equations*

$$x \equiv 0 (\text{mod } d_1), \; x \equiv 2 (\text{mod } d_2)$$

Equivalently we have proved the number of upper primes of a twin prime pair between p_θ and N is given by $T(p_\theta, N, 2)$ as defined in the above theorem.

Chapter 4

The Twin Primes Theorem

We have,

$$\begin{aligned} T(p_\theta, N, 2) &= \sum^{*} \mu(d_1 d_2) \left[\frac{N - x_1}{d_1 d_2} \right] \\ &= \sum^{*} \mu(d_1 d_2) \left(\frac{N - x_1 - x_2}{d_1 d_2} \right) \end{aligned}$$

where x_1 is the least non-negative solution of the system of linear congruences,

$$x \equiv 0 (\bmod\ d_1) \text{ and } x \equiv 2 (\bmod\ d_2)$$

and x_2 is the least non-negative solution of the linear congruence,

$$x_2 \equiv (N - x_1)(\bmod\ d_1 d_2)$$

Lemma 13.
x_2 is the least non-negative solution of the system of linear congruences,

$$x \equiv N (\bmod\ d_1) \textit{ and } x \equiv (N-2)(\bmod\ d_2)$$

Proof.

$$x_2 \equiv (N - x_1)(\bmod\ d_1 d_2) \Rightarrow x_2 = N - x_1 + k d_1 d_2, \ k \in \mathbb{Z}$$

But also,

$$\begin{aligned} x_1 \equiv 0 (\bmod\ d_1) &\Rightarrow x_1 = a d_1, a \in \mathbb{Z} \\ &\Rightarrow x_2 = N - a d_1 + k d_1 d_2 \\ &\Rightarrow x_2 \equiv N (\bmod\ d_1) \end{aligned}$$

Then $x_2 = N - x_1 + k d_1 d_2 \Rightarrow x_1 = N - x_2 + k d_1 d_2$ gives,

$$\begin{aligned} & x_1 \equiv (N - x_2)(\bmod\ d_1 d_2) \text{ and also } x_1 \equiv 2 (\bmod\ d_2) \\ & \Rightarrow 2 + a d_2 = N - x_2 + k d_1 d_2, \ a, k \in \mathbb{Z} \\ & \Rightarrow x_2 \equiv (N - 2)(\bmod\ d_2) \end{aligned}$$

So x_2 is the solution of the pair of linear congruences,

$$x \equiv N(\text{mod } d_1) \text{ and } x \equiv (N-2)(\text{mod } d_2)$$

Since $x_2 < d_1 d_2$ then x_2 is the least non-negative solution of these two congruences. □

Note this means x_2 values can be viewed as independent of the x_1 values, depending only on N and the sieving primes.

Theorem 14. *Twin Primes Theorem*
There are an infinite number of twin prime pairs.

Proof. Suppose (p, q) is the largest twin prime pair with $q - p = 2$ and p, q both primes. Suppose r is the next highest prime. r exists because there are an infinite number of primes and indeed, by Bertrand's Postulate, r lies between q and $2q$. Note, given the value of the largest known twin prime pair, $r^2 - q^2 = (r+q)(r-q)$ is a very large number.
Choose $N = q^2 + 1$ and $\overline{N} = r^2 - 1$.
Then q is the largest prime less than the square root of both $q^2 + 1$ and $r^2 - 1$.
Hence, using Theorem 12, and noting 1 is always counted,

$$T(q, q^2+1, 2) = \sum^{*} \mu(d_1 d_2) \frac{q^2 + 1 - x_1 - x_2}{d_1 d_2} = 1 \tag{4.0.1}$$

where x_1 is the least non-negative solution of the system of congruences

$$x_1 \equiv 0(\text{mod } d_1) \text{ and } x_1 \equiv 2(\text{mod } d_2) \tag{4.0.2}$$

and x_2 is the least non-negative solution of the system of congruences

$$x_2 \equiv q^2 + 1(\text{mod } d_1) \text{ and } x_2 \equiv q^2 - 1(\text{mod } d_2)$$

Also, again using Theorem 12,

$$T(q, r^2-1, 2) = \sum^{*} \mu(d_1 d_2) \frac{r^2 - 1 - \overline{x_1} - \overline{x_2}}{d_1 d_2} = 1 \tag{4.0.3}$$

where $\overline{x_1}$ is the least non-negative solution of the system of congruences

$$\overline{x_1} \equiv 0(\text{mod } d_1) \text{ and } \overline{x_1} \equiv 2(\text{mod } d_2) \tag{4.0.4}$$

and $\overline{x_2}$ is the least non-negative solution of the system of congruences

$$x \equiv r^2 - 1(\text{mod } d_1) \text{ and } x \equiv r^2 - 3(\text{mod } d_2)$$

Subtracting (4.0.3) - (4.0.1) gives,

$$\sum^{*} \mu(d_1 d_2) \frac{r^2 - q^2 - 2 + x_1 - \overline{x_1} + x_2 - \overline{x_2}}{d_1 d_2} = 0 \tag{4.0.5}$$

By (4.0.2) and (4.0.4) we have $x_1 = \overline{x_1}$ for each respective $d_1 d_2$, so we have from (4.0.5),

$$\overline{x_2} - x_2 = r^2 - q^2 - 2 \tag{4.0.6}$$

for all choices of $d_1 d_2$. Note $r^2 - q^2 - 2$ is a fixed number.
But, by Lemma 13, for each choice of (d_1, d_2), $\overline{x_2}$ is the least non-negative solution of the system,

$$\overline{x_2} \equiv r^2 - 1 (\text{mod } d_1), \ \overline{x_2} \equiv r^2 - 3 (\text{mod } d_2)$$

and x_2 is the least non-negative solution of the system,

$$x_2 \equiv q^2 + 1 (\text{mod } d_1), \ x_2 \equiv q^2 - 1 (\text{mod } d_2)$$

Hence, subtracting,

$$\overline{x_2} - x_2 \equiv r^2 - q^2 - 2 (\text{mod } d_1), \ \overline{x_2} - x_1 \equiv r^2 - q^2 - 2 (\text{mod } d_2)$$

making,

$$\overline{x_2} - x_2 \equiv r^2 - q^2 - 2 (\text{mod } d_1 d_2) \tag{4.0.7}$$

Hence it is not possible, as required by (4.0.6), to have $\overline{x_2} - x_2 = r^2 - q^2 - 2$ for all choices of $d_1 d_2$, for example $d_1 d_2 = 3 * 5$ where, by (4.0.7), $\overline{x_2} - x_2 < 15$.
Therefore the assumption (p, q) is the largest twin prime pair is false. We conclude there are an infinite number of twin prime pairs.

$\square$